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Matrix Games under an Intuitionistic Fuzzy Duality Approach with Exponential Functions: MRI-Related Cochlear Implant Analysis

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ABSTRACT

Game theory has evolved to incorporate various forms of uncertainty. The intuitionistic fuzzy (I-Fuzzy) framework is a powerful tool for modeling such uncertainties, particularly in two-person zero-sum matrix games (TZMGs) with imprecise goals. This study addresses TZMGs where goals are represented in I-Fuzzy form under pessimistic, optimistic, and mixed approaches. By applying exponential functions, optimization problems are constructed for each player. These problems are then transformed into crisp linear programming problems (LPPs) using logarithmic functions, enabling the derivation of optimal strategies. The role and impact of the shape parameter in exponential functions are analyzed, highlighting its influence through comparative insights. To illustrate the effectiveness and real-world relevance of the proposed approach, a practical healthcare problem evaluating adverse MRI effects on cochlear implant (CI) users is solved, showcasing both the methodology's applicability and its potential for decision-making under uncertainty.

1. Introduction

Game theory is a mathematical study used to analyze the strategic interactions between two or more than two rational decision makers (players). The fundamentals of game theory were first presented in German language by Neumann in the research "Zur Theorie Der Gesellschaftsspiele (1928)" [**1**]. The english extension of this German research and the fundamentals of game theory are presented in the book "Theory of Games and Economic Behaviour (1944)" by Neumann and Morgenstern [**2**]. As per game theory, a finite number of strategies is defined for each player and all players utilize their

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respective strategies one after other. A distinct payoff is specified to each combination of strategic interactions between the players. Game theory provides tools to predict the outcomes of these interactions, assuming that the framework of the game is structured in an unbiased manner for each player and each player seeks to maximize their own payoffs. Non-cooperative games is a high application classification of games defined under game theory with an essential condition that the players cannot hold on any pre-play communication. TZMG is a categorization of non-cooperative games involving two players in which the game is structured in a manner that the gain of one player is equal to other player's loss. Game theory is applicable in numerous domains such as politics, commercial and business operations, field and board games, military and civil defense, engineering applications, robotics, multi-agent systems, networks etc [**3**].

In idealist situations, it is assumed that the goals of TZMG under crisp scenario are precisely defined but in practicality, various factors associated with the game problem makes it difficult to establish more precise results. These factors include unpredictability of events, incomplete information, vagueness within given data etc. Fuzzy set theory [**4**] is an inerrant tool to deal with the discrepancy within the given data. Fuzzy environment, by the utilization of membership function has been widely used to cater with the vague nature of goals in TZMGs. Fuzzy environment incorporated within TZMGs advances the study in a more realistic manner. Unlike the fuzzy environment, I-Fuzzy environment acts as a more scrupulous medium in the study of TZMGs as it caters the nuance of indeterminacy within the goals of the TZMGs by the use of a non-membership function along with a membership function. The I-Fuzzy environment makes use of two major categorization of membership and non-membership functions : linear (triangular, trapezoidal etc) and non-linear (exponential, logarithmic etc). The non-linear membership functions because of their explicit representation are considered as a more viable tool in comparison to the linear membership functions which shortfall for a meticulous depiction of the problem.

1.1 *Literature review*

Campos [**5**] was among the first to investigate a TZMG with imprecise payoff matrices and introduced a generalized approach to extend the classical solution method for matrix games. Atanassov [**6**] proposed a new form of fuzzy sets called I-Fuzzy sets. Bector et al. [**7**] showed that duality in fuzzy linear programming is equivalent to TZMG with fuzzy payoffs, with limitations noted in earlier studies. Aggarwal et al. [**8**] extended TZMG from a fuzzy framework to an I-Fuzzy framework and proposed a new solution concept to address indeterminacy in players' aspiration levels. Aggarwal et al. [**9**] presented an approach providing efficient solution to a class of I-Fuzzy matrix games with piecewise linear membership and non-membership functions. Kumar [**10**] proposed a max-min solution approach to solve multi-objective matrix games with fuzzy goals. Khan et al. [**11**] solved matrix games with Atanassov's I-Fuzzy goals via indeterminacy resolution approach. Kumar [**12**] established the similarities between I-Fuzzy programming and goal programming. Debnath and Gupta [**13**] analyzed I-Fuzzy linear primal-dual problems using exponential membership functions and showed improved duality performance over linear membership functions. Kumar [**14**] presented a maxmin–minmax solution to multi-objective TZMGs with I-Fuzzy goals. Zheng and Brikaa [**15**] proposed an aspiration-level-based approach for solving multi-objective bi-matrix games with I-Fuzzy goals, extending the classical fuzzy framework. Naqvi et al. [**16**] studied matrix games with qualitative payoffs modeled by linguistic interval-valued intuitionistic fuzzy numbers and derived solutions via equivalent linear and nonlinear programming formulations. Seikh and Dutta [**17**] developed a method for solving matrix games with picture fuzzy payoffs by employing multi-objective nonlinear programming and a weighted average approach. Fujita et al. [**18**] introduced the notion of the hyperfuzzy offgraph by incorporating various uncertainity behaviors on both vertices and edges. Kumar and Garg [**19**] proposed a two-level fuzzy framework for solving single- and multi-objective matrix games with fuzzy goals and payoffs. Kumar and Aashish [**20**] utilized Archimedean t-norm and t-conorm based approach to solve q-rung orthopair fuzzy matrix games and backed its efficacy with application to electric vehicle market share problem. Majid et al. [**21**] utilized content analysis and the Fermatean Fuzzy TOPSIS method to rank strategies based on business intelligence in the context of smart cities. Malik and Gupta [**22**] developed an I-Fuzzy multi-objective linear fractional programming model, employing linearized formulations and membership-based approaches to handle uncertainty and conflicting objectives.

1.2 *Motivation and the Present Research Work*

Upon reviewing the aforementioned literature, it is evident that exponential membership and non-membership functions capture vagueness and ambiguity in fuzzy optimization problems more effectively than their linear counterparts [**13**, **22**, **23**]. For supporting the logic to utilize the exponential membership and non-membership functions in fuzzy game modelling, the following arguments are presented:

$\mathcal{A}1$ In data analysis, while training algorithms or performing gradient-based optimization, the mathematical functions must be differentiable everywhere. Linear functions gave sharp corners at their peaks and bases which causes gradients to be undefined or jump abruptly, which can destabilize optimization algorithms. Exponential functions (e.g., Gaussian-like or Sigmoidal) on the other hand are smooth and differentiable across their entire domain. This allows for stable, continuous gradient descent, leading to more accurate model convergence.

$\mathcal{A}2$ Real-world data rarely follows strict linear progressions. Human perception of vague concepts (“young”, “hot”, “risky”) tends to follow non-linear sensitivity. Exponential functions decay or grow rapidly at first and then plateau or vice versa. They capture the “curved” nature of real-world distributions better than straight lines, reducing the information loss that occurs when we tweak complex data into linear shapes.

$\mathcal{A}3$ In IFSs, uncertainity is measured by the criteria of hesitation margin also. Exponential functions while calculating the entropy of a dataset often provide a more sensitive measure of information content, helping algorithms distinguish between “slightly fuzzy” and “very fuzzy” data points more effectively than linear approximations. Further, exponential functions are closely linked to the principle of maximum entropy similar to Gaussian distributions which aim to maximize entropy for a given variance.

$\mathcal{A}4$ In an intuitionistic environment, the sum of membership (μ) and non-membership (ν) must be less than or equal to 1. Linear functions, following to the constraint while maintaining independent shapes for μ and ν can be geometrically awkward. Exponential forms allow for peculiar definitions where μ and ν interact smoothly. For example, if μ is defined exponentially, ν can be defined as a complementary exponential function that naturally respects the constraint while allowing the hesitation to vary dynamically across the domain.

Inspired by this concept, we explore a TZMG with Intuitionistic Fuzzy Goals (TMGIG), where the membership and non-membership functions of the goals are modeled using triangular exponential forms. The resulting optimization problems for both players take the form of two nonlinear programming models that are fuzzy duals of each other. To illustrate the influence of the shape parameter on the minimal degree of acceptability and maximal degree of rejection, a numerical example is presented under the pessimistic scenario. To validate the results obtained from the proposed models, two comparative analyses have also been conducted against existing approaches.

The present study provides the following contributions:

$C1$ To formulate a TMGIG, the goals' membership and non-membership functions are represented using exponential expressions. Additionally, the effect of shape parameters on the minimal degree of acceptability and the maximal degree of rejection is analyzed.

$C2$ To show the relevance and effectiveness, a numerical example is solved with a comparative analysis over the existing work.

$C3$ To show the real-life applicability and feasibility, a problem of healthcare management system associated with adverse effects of magnetic resonance imaging (MRI) with cochlear implants (CIs) has been solved. The role of shape parameter is also discussed.

The outlines of the remaining paper are given as follows. In Section 2, we present the I-Fuzzy duality results by exponential membership and non-membership functions. In Section 3, we give the formulation and solution of TZMG with I-Fuzzy goals expressed in the form of exponential membership and non-membership functions. To illustrate the feasibility of the proposed solution procedure, a numerical example has been solved in Section 4. In Subsection 4.1, the effectiveness and reliability of the proposed approach are rigorously examined through two detailed comparative analyses with established methods. To demonstrate the real-life applicability of the proposed solution approach, a healthcare management system problem to analyze the MRI associated CI adverse effects has been considered in Section 5. Finally, in Section 6, the conclusion from our findings and a brief discussion on the future scope of this topic are discussed.

2. I-Fuzzy Duality for Exponential Goal Functions

The I-Fuzzy duality results corresponding to exponential membership and non-membership functions are presented as follows.

The fuzzy linear programming Primal-Dual problems (**M1**) & (**M2**) can be defined as:

$$\textbf{(M1)} \qquad \text{Find} \quad x \in \mathbb{R}^n$$
$$\text{s.t.} \quad \begin{cases} c^T x \gtrsim u_0, \\ Ax \lesssim b, \\ x \geq 0. \end{cases}$$

$$\textbf{(M2)} \qquad \text{Find} \quad y \in \mathbb{R}^m$$
$$\text{s.t.} \quad \begin{cases} b^T y \lesssim v_0, \\ A^T y \gtrsim c, \\ y \geq 0. \end{cases}$$

Here $\mathbb{R}^n$ is the n-dimensional Euclidean space and $\mathbb{R}^m$ represents the m-dimens-ional Euclidean space such that c, x $\in \mathbb{R}^n$, b, y $\in \mathbb{R}^m$ and A is a $(m \times n)$ real-matrix. The symbol $\gtrsim$ means essentially greater than or equal, symbol $\lesssim$ denotes essentially less than or equal, u_0 denotes the aspiration level for primal objective in **(M1)** and v_0 denotes the aspiration level for dual objective in **(M2)**.

The Primal-Dual problems **(M1)** and **(M2)** can be extended to I-Fuzzy Primal-Dual problems ($\overline{\textbf{M1}}$) and ($\overline{\textbf{M2}}$) as follows:

$$(\overline{\textbf{M1}}) \qquad \text{Find} \quad x \in \mathbb{R}^n$$
$$\text{s.t.}$$
$$\begin{cases} c^T x \gtrsim_{IF} u_0, \\ Ax \lesssim_{IF} b, \\ x \geq 0. \end{cases}$$

$$(\overline{\textbf{M2}}) \qquad \text{Find} \quad y \in \mathbb{R}^m$$
$$\text{s.t.}$$
$$\begin{cases} b^T y \lesssim_{IF} v_0, \\ A^T y \gtrsim_{IF} c, \\ y \geq 0, \end{cases}$$

where "$\gtrsim_{IF}$" and "$\lesssim_{IF}$" are termed as "essentially greater than or equal to" and "essentially less than or equal to" in I-Fuzzy sense [**8**], respectively.

The I-Fuzzy problems with membership and non-membership functions for given aspirational values and their respective tolerances can be classified into three categories based on the will of the decision maker as in which manner he/she wishes to implement his/her strategy to solve the problem. This leads us to the classification of exponential membership and non-membership functions defined for I-Fuzzy primal-dual problems based on the approaches (optimistic, pessimistic and mixed) of the player.

2.1 *Pessimistic Approach*

In the pessimistic approach, the decision maker tends to exhibit a greater inclination toward rejection while accepting less than usual. To effectively capture the levels of satisfaction and dissatisfaction under this mindset-both for the objective function and the set of constraints-we adopt the following form of exponential membership and non-membership functions:

For ($\overline{\mathbf{M1}}$)

$$\mu_E^P(c^T x) = \begin{cases} 1, & c^T x \geq u_0, \\ \frac{e^{-\alpha_0\left(\frac{u_0 - c^T x}{p_0}\right)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}}, & u_0 - p_0 \leq c^T x \leq u_0, \\ 0, & c^T x \leq u_0 - p_0. \end{cases},$$

$$\mu_E^P(A_i x) = \begin{cases} 1, & A_i x \leq b_i, \\ \frac{e^{-\alpha_i\left(\frac{A_i x - b_i}{p_i}\right)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}}, & b_i \leq A_i x \leq b_i + p_i, \\ 0, & A_i x \geq b_i + p_i. \end{cases},$$

and

$$\nu_E^P(c^T x) = \begin{cases} 1, & c^T x \leq u_0 - p_0, \\ \frac{e^{-\alpha_0\left(\frac{c^T x - u_0 + p_0}{q_0}\right)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}}, & u_0 - p_0 \leq c^T x \leq u_0 - p_0 + q_0, \\ 0, & c^T x \geq u_0 - p_0 + q_0. \end{cases},$$

$$\nu_E^P(A_i x) = \begin{cases} 1, & A_i x \geq b_i + p_i, \\ \frac{e^{-\alpha_i\left(\frac{b_i - A_i x + p_i}{q_i}\right)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}}, & b_i + p_i - q_i \leq A_i x \leq b_i + p_i, \\ 0, & A_i x \leq b_i + p_i - q_i. \end{cases},$$

where $i = 1, 2, \ldots, m$, the parameters α_0, α_i, where $0 < \alpha_0$, $\alpha_i < \infty$, are referred as shape parameters that reflect the degree of vagueness. The parameters $p_0 > 0$ and $q_0 > 0$ represents the tolerances associated with the membership and non-membership function of the objective function, respectively, with the condition $0 < q_0 < p_0$. Similarly, for the i^{th} constraint, $p_i > 0$ and $q_i > 0$ denote the tolerances related to the membership and non-membership functions, respectively, satisfying $0 < q_i < p_i$. Here, A_i, b_i represent the i^{th} row of the matrix A and i^{th} element of the vector b, respectively, for $i = 1, 2, \ldots, m$.

For ($\overline{\mathbf{M2}}$)

$$\mu_E^P(b^T y) = \begin{cases} 1, & b^T y \leq v_0, \\ \frac{e^{-\beta_0\left(\frac{b^T y - v_0}{s_0}\right)} - e^{-\beta_0}}{1 - e^{-\beta_0}}, & v_0 \leq b^T y \leq v_0 + s_0, \\ 0, & b^T y \geq v_0 + s_0. \end{cases},$$

$$\mu_E^P(A_j^T y) = \begin{cases} 1, & A_j^T y \geq c_j, \\ \frac{e^{-\beta_j\left(\frac{c_j - A_j^T y}{s_j}\right)} - e^{-\beta_j}}{1 - e^{-\beta_j}}, & c_j - s_j \leq A_j^T y \leq c_j, \\ 0, & A_j^T y \leq c_j - s_j. \end{cases},$$

and

$$\nu_E^P(b^T y) = \begin{cases} 1, & b^T y \geq v_0 + s_0, \\ \frac{e^{-\beta_0\left(\frac{v_0 - b^T y + s_0}{t_0}\right)} - e^{-\beta_0}}{1 - e^{-\beta_0}}, & v_0 + s_0 - t_0 \leq b^T y \leq v_0 + s_0, \\ 0, & b^T y \leq v_0 + s_0 - t_0. \end{cases},$$

$$\nu_E^P(A_j^T y) = \begin{cases} 1, & A_j^T y \leq c_j - s_j, \\ \frac{e^{-\beta_j\left(\frac{A_j^T y + s_j - c_j}{t_j}\right)} - e^{-\beta_j}}{1 - e^{-\beta_j}}, & c_j - s_j \leq A_j^T y \leq c_j - s_j + t_j, \\ 0, & A_j^T y \geq c_j - s_j + t_j. \end{cases},$$

where $j = 1, 2, \ldots, n$, the parameters β_0, β_j, where $0 < \beta_0, \beta_j < \infty$, are referred as shape parameters that reflect the degree of vagueness. The parameters $s_0 > 0$ and $t_0 > 0$ represents the tolerances associated with the membership and non-membership function of the objective function, respectively, with the condition $0 < t_0 < s_0$. Similarly, for the j^{th} constraint, $s_j > 0$ and $t_j > 0$ denote the tolerances related to the membership and non-membership functions, respectively, satisfying $0 < t_j < s_j$. Here, A_j, c_j represent the j^{th} column of the matrix A and j^{th} element of the vector c, respectively, for $j = 1, 2, \ldots, n$.

In the aforementioned exponential membership and non-membership functions the shape parameters α_0, α_i, β_0 and β_j act as paramount factors in anlayzing the optimal results. The prevalence and sensitivity to the optimal results of these parameters are described as follows:

(i) **Description:** In an IFS, particularly utilizing exponential membership and non-membership functions, the shape parameter acts as a critical turning knob. It dictates how strictly or loosely, the fuzzy set interprets input data. Exponential forms introduces non-linearity, making the shape parameter essential for modelling complex real-world data.

(ii) **Advantages:** Shape parameter defines the curvature or steepness of the function, directly influencing how data is mapped to fuzzy values. If data is dense and requires fine distinctions between points (e.g., precision engineering measurements), a higher shape parameter helps seperate them. If the data is sparse or inherently vague (e.g., linguistic opinions like "good" vs "fair"), a lower shape parameter smoothens the transitions. Mentioned below, are the advantages of shape parameter in the analysis of fuzzy data:

   - It allows a single mathematical model to adapt to different types of uncertainity without changing the fundamental problem.
   - By adjusting the shape, the effect of outliers can be dampened. A flatter curve having lower shape parameter value makes the model less sensitive to minor data fluctuations.
   - Many real-world phenomena (risk assessment, adverse drug reactions etc.) are not linear. The exponential shape parameter models these exponential decays or growths more accurately than linear membership function : triangular or trapezoidal.

(iii) **Sensitivity Analysis and The Shape Parameter :** Sensitivity analysis in the context of shape parameter is the process of varying the shape parameter to observe how the final output changes. This is crucial for validating the robustness of a model. Here are the key facts regarding sensitivity results due to changes in the shape parameter:

   - If the ranking of alternatives remains constant across a wide range of shape parameter values, the decision is considered robust. This indicates that the best choice is clearly superior, regardless of how strictly or loosely you define the criteria.
   - As the shape parameter monotonically increases or decrease, the shape parameter reach a threshold where the ranking of two alternatives flips. This reveals that the superiority of any one of the alternatives depends on a specific interpretation of uncertainity and is context-dependent.
   - The sensitivity analysis mathematically mirrors the psychological attitude of the decision-maker. A pessimistic decision maker might prefer a shape parameter that yields lower membership values for the same data. An optimistic decision-maker might prefer a parameter that yields higher membership. Sensitivity analysis helps visualize how "optimisim" or "pessimism" alters the final outcome.

- Changing the shape parameter affects the entropy of the system. In an I-Fuzzy environment, extreme values of the shape parameter might force the hesitiation index to shrink or expand. Sensitivity analysis ensures that the chosen parameter does not artificially eliminate the hesitation inherent in the problem.

Let σ_1^P and σ_2^P represent the minimal degree of acceptance and maximal degree of rejection, respectively, for the problem ($\overline{\textbf{M1}}$). In line with Angelov's approach [**24**], and by incorporating exponential membership and non-membership functions within the pessimistic framework, the problem ($\overline{\textbf{M1}}$) can be transformed into the following crisp optimization problem:

$$(\textbf{PP} - \overline{\textbf{M1}}) \quad \max \quad \sigma_1^P - \sigma_2^P$$
$$\text{s.t.}$$
$$\begin{cases} \sigma_1^P \leq \dfrac{e^{-\alpha_0\left(\frac{u_0 - c^T x}{p_0}\right)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}} = \dfrac{e^{\alpha_0\left(\frac{c^T x - u_0}{p_0}\right)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}}, \\ \sigma_2^P \geq \dfrac{e^{-\alpha_0\left(\frac{c^T x - u_0 + p_0}{q_0}\right)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}} = \dfrac{e^{\alpha_0\left(\frac{u_0 - p_0 - c^T x}{q_0}\right)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}}, \\ \sigma_1^P \leq \dfrac{e^{-\alpha_i\left(\frac{A_i x - b_i}{p_i}\right)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}} = \dfrac{e^{\alpha_i\left(\frac{b_i - A_i x}{p_i}\right)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}}; \qquad (i = 1, 2, \ldots, m), \\ \sigma_2^P \geq \dfrac{e^{-\alpha_i\left(\frac{b_i - A_i x + p_i}{q_i}\right)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}} = \dfrac{e^{\alpha_i\left(\frac{A_i x - b_i - p_i}{q_i}\right)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}}; \quad (i = 1, 2, \ldots, m), \\ 0 \leq \sigma_2^P \leq \sigma_1^P \leq 1, \\ \sigma_1^P + \sigma_2^P \leq 1, \\ x \geq 0. \end{cases}$$

The problem (**PP**-$\overline{\textbf{M1}}$) can be re-written as:

$$(\textbf{PP} - \overline{\textbf{M1}}) \quad \max \quad \sigma_1^P - \sigma_2^P$$
$$\text{s.t.}$$
$$\begin{cases} p_0 \ln\left(\sigma_1^P (1 - e^{-\alpha_0}) + e^{-\alpha_0}\right) \leq \alpha_0 \left(c^T x - u_0\right), \\ q_0 \ln\left(\sigma_2^P (1 - e^{-\alpha_0}) + e^{-\alpha_0}\right) \geq \alpha_0 \left(u_0 - p_0 - c^T x\right), \\ p_i \ln\left(\sigma_1^P (1 - e^{-\alpha_i}) + e^{-\alpha_i}\right) \leq \alpha_i (b_i - A_i x); \qquad (i = 1, 2, \ldots, m), \\ q_i \ln\left(\sigma_2^P (1 - e^{-\alpha_i}) + e^{-\alpha_i}\right) \geq \alpha_i (A_i x - b_i - p_i); \quad (i = 1, 2, \ldots, m), \\ 0 \leq \sigma_2^P \leq \sigma_1^P \leq 1, \\ \sigma_1^P + \sigma_2^P \leq 1, \\ x \geq 0. \end{cases}$$

Let χ_1^P and χ_2^P represent the minimal degree of acceptance and maximal degree of rejection, respectively, for the problem ($\overline{\textbf{M2}}$). In line with Angelov's approach [**24**], and by incorporating exponential membership and non-membership functions within the pessimistic framework, the problem ($\overline{\textbf{M2}}$)

can be transformed into the following crisp optimization problem:

$$(\mathbf{PD}-\overline{\mathbf{M2}})\quad \max \quad \chi_1^P-\chi_2^P$$
$$\text{s.t.}$$
$$\begin{cases}
\chi_1^P \leq \dfrac{e^{-\beta_0\left(\frac{b^Ty-v_0}{s_0}\right)}-e^{-\beta_0}}{1-e^{-\beta_0}}=\dfrac{e^{\beta_0\left(\frac{v_0-b^Ty}{s_0}\right)}-e^{-\beta_0}}{1-e^{-\beta_0}},\\
\chi_2^P \geq \dfrac{e^{-\beta_0\left(\frac{v_0-b^Ty+s_0}{t_0}\right)}-e^{-\beta_0}}{1-e^{-\beta_0}}=\dfrac{e^{\beta_0\left(\frac{b^Ty-v_0-s_0}{t_0}\right)}-e^{-\beta_0}}{1-e^{-\beta_0}},\\
\chi_1^P \leq \dfrac{e^{-\beta_j\left(\frac{c_j-A_j^Ty}{s_j}\right)}-e^{-\beta_j}}{1-e^{-\beta_j}}=\dfrac{e^{\beta_j\left(\frac{A_j^Ty-c_j}{s_j}\right)}-e^{-\beta_j}}{1-e^{-\beta_j}}; \qquad (j=1,2,\dots,n),\\
\chi_2^P \geq \dfrac{e^{-\beta_j\left(\frac{A_j^Ty+s_j-c_j}{t_j}\right)}-e^{-\beta_j}}{1-e^{-\beta_j}}=\dfrac{e^{\beta_j\left(\frac{c_j-s_j-A_j^Ty}{t_j}\right)}-e^{-\beta_j}}{1-e^{-\beta_j}}; \quad (j=1,2,\dots,n),\\
0\leq \chi_2^P\leq \chi_1^P\leq 1,\\
\chi_1^P+\chi_2^P\leq 1,\\
y\geq 0.
\end{cases}$$

The problem (**PD-$\overline{\mathbf{M2}}$**) can be re-written as:

$$(\mathbf{PD}-\overline{\mathbf{M2}})\quad \max \quad \chi_1^P-\chi_2^P$$
$$\text{s.t.}$$
$$\begin{cases}
s_0\ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right)\leq \beta_0\left(v_0-b^Ty\right),\\
t_0\ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right)\geq \beta_0\left(b^Ty-v_0-s_0\right),\\
s_j\ln\left(\chi_1^P\left(1-e^{-\beta_j}\right)+e^{-\beta_j}\right)\leq \beta_j\left(A_j^Ty-c_j\right); \qquad (j=1,2,\dots,n),\\
t_j\ln\left(\chi_2^P\left(1-e^{-\beta_j}\right)+e^{-\beta_j}\right)\geq \beta_j\left(c_j-s_j-A_j^Ty\right); \quad (j=1,2,\dots,n),\\
0\leq \chi_2^P\leq \chi_1^P\leq 1,\\
\chi_1^P+\chi_2^P\leq 1,\\
y\geq 0.
\end{cases}$$

Similar results can be formulated for optimistic and mixed cases. Using these results, we now model a TZMG with I-Fuzzy information in the next section.

3. Solution of TZMGs with I-Fuzzy Goals

Let A be a $(m\times n)$ real matrix. Suppose $\mathbb{U}^m=\{x\in\mathbb{R}^m|\ x\geq 0$ and $\sum_{i=1}^m x_i=1\}$ and $\mathbb{U}^n=\{y\in\mathbb{R}^n|\ y\geq 0$ and $\sum_{j=1}^n y_j=1\}$ be the strategy space for player-I ($\mathbb{K}1$) and player-II ($\mathbb{K}2$), respectively. Let u_0 and v_0 be the aspiration levels for $\mathbb{K}1$ and $\mathbb{K}2$, respectively. Let p_0, q_0 and s_0, t_0 be the tolerances of acceptance, rejection associated with the aspiration levels u_0 and v_0 of $\mathbb{K}1$ and $\mathbb{K}2$, respectively. Then, $TMGIG$ is expressed as:

$$TMGIG=(\mathbb{U}^m,\ \mathbb{U}^n,\ A;\ u_0,\ \gtrsim_{IF},\ p_0,\ q_0;\ v_0,\ \lesssim_{IF},\ s_0,\ t_0)$$

where $\gtrsim_{IF}$ and $\lesssim_{IF}$ are termed as "essentially greater than or equal to" and "essentially less than or equal to", in I-Fuzzy sense [**8**], respectively.

Let $x^T Ay$ is the expected payoff of $\mathbb{K}1$, and as the $TMGIG$ is zero-sum the expected payoff of $\mathbb{K}2$ is $-x^T Ay$.

A point $(\bar{x}, \bar{y}) \in \mathbb{U}^m \times \mathbb{U}^n$ is called a solution of the $TMGIG$ if

1. $(\bar{x})^T Ay \;\; \gtrsim_{IF(p_0,q_0)} \;\; u_0 \;\; \forall\, y \in \mathbb{U}^n$,
2. $x^T A(\bar{y}) \;\; \lesssim_{IF(s_0,t_0)} \;\; v_0 \;\; \forall\, x \in \mathbb{U}^m$.

where "$\gtrsim_{IF(p_0,q_0)}$" represents "essentially greater than or equal to in I-Fuzzy sense with tolerances p_0 and q_0". Similarly, "$\lesssim_{IF(s_0,t_0)}$" represents "essentially less than or equal to in I-Fuzzy sense with tolerances s_0 and t_0".

According to [**8**], the $TMGIG$ is viewed as two optimization problems. These optimization problems are equivalent to two I-Fuzzy LPPs which are given as:

$$(\overline{\textbf{N1}}) \quad \text{Find x} \in \mathbb{U}^m \quad \text{such that} \quad \sum_{i=1}^{m} a_{ij}x_i \gtrsim_{IF(p_0,q_0)} u_0, \quad j = 1, 2, \ldots, n.$$

$$(\overline{\textbf{N2}}) \quad \text{Find y} \in \mathbb{U}^n \quad \text{such that} \quad \sum_{j=1}^{n} a_{ij}y_j \lesssim_{IF(s_0,t_0)} v_0, \quad i = 1, 2, \ldots, m.$$

LPPs associated with the $TMGIG$, incorporating exponential membership and non-membership functions for specified aspiration levels and their corresponding tolerances, can be grouped into three distinct categories. These categories reflect the player's preference regarding how they intend to pursue their strategy for solving the problem:

(i) Pessimistic Approach

(ii) Optimistic Approach

(iii) Mixed Approach

In pessimistic approach, the membership and non-membership functions for the goals are defined as outlined in Section 2.

Hence, we consider only pessimistic approach for solving the present game problem.

3.1 LPPs for $TMGIG$ under Pessimistic Approach

Let σ_1^P and σ_2^P represent the minimal degree of acceptance and the maximal degree of rejection, respectively, for the problem ($\overline{\textbf{N1}}$). In accordance with the methodology proposed by Angelov [**24**], and by employing exponential membership and non-membership functions under the pessimistic approach, the problem ($\overline{\textbf{N1}}$) can be transformed into the following crisp optimization model.

$$(\mathbf{CP}-\overline{\mathbf{N1}})\quad \max \quad \sigma_1^P-\sigma_2^P$$
$$\text{s.t.}$$
$$\begin{cases} \sigma_1^P \le \dfrac{e^{-\alpha_0\left(\frac{u_0-A_j^T x}{p_0}\right)}-e^{-\alpha_0}}{1-e^{-\alpha_0}} = \dfrac{e^{\alpha_0\left(\frac{A_j^T x-u_0}{p_0}\right)}-e^{-\alpha_0}}{1-e^{-\alpha_0}}; \qquad (j=1,2,\ldots,n),\\ \sigma_2^P \ge \dfrac{e^{-\alpha_0\left(\frac{A_j^T x-u_0+p_0}{q_0}\right)}-e^{-\alpha_0}}{1-e^{-\alpha_0}} = \dfrac{e^{\alpha_0\left(\frac{u_0-p_0-A_j^T x}{q_0}\right)}-e^{-\alpha_0}}{1-e^{-\alpha_0}};\ (j=1,2,\ldots,n),\\ 0\le \sigma_2^P \le \sigma_1^P \le 1,\\ \sigma_1^P+\sigma_2^P\le 1,\\ \sum_{i=1}^m x_i = 1,\\ x\ge 0. \end{cases}$$

The problem (**CP**-$\overline{\mathbf{N1}}$) can be re-written as:

$$(\mathbf{CP}-\overline{\mathbf{N1}})\quad \max \quad \sigma_1^P-\sigma_2^P$$
$$\text{s.t.}$$
$$\begin{cases} p_0\ln\left(\sigma_1^P\left(1-e^{-\alpha_0}\right)+e^{-\alpha_0}\right)\le \alpha_0\left(A_j^T x-u_0\right); \qquad (j=1,2,\ldots,n),\\ q_0\ln\left(\sigma_2^P\left(1-e^{-\alpha_0}\right)+e^{-\alpha_0}\right)\ge \alpha_0\left(u_0-p_0-A_j^T x\right);\ (j=1,2,\ldots,n),\\ 0\le \sigma_2^P \le \sigma_1^P \le 1,\\ \sigma_1^P+\sigma_2^P\le 1,\\ \sum_{i=1}^m x_i = 1,\\ x\ge 0. \end{cases}$$

Let χ_1^P and χ_2^P represent the minimal degree of acceptance and the maximal degree of rejection, respectively, for the problem ($\overline{\mathbf{N2}}$). In accordance with the methodology proposed by Angelov [**24**], and by employing exponential membership and non-membership functions under the pessimistic approach, the problem ($\overline{\mathbf{N2}}$) can be transformed into the following crisp optimization model.

$$(\mathbf{CP}-\overline{\mathbf{N2}})\quad \max \quad \chi_1^P-\chi_2^P$$
$$\text{s.t.}$$
$$\begin{cases} s_0\ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right)\le \beta_0\left(v_0-A_i y\right); \qquad (i=1,2,\ldots,m),\\ t_0\ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right)\ge \beta_0\left(A_i y-v_0-s_0\right);\ (i=1,2,\ldots,m),\\ 0\le \chi_2^P \le \chi_1^P \le 1,\\ \chi_1^P+\chi_2^P\le 1,\\ \sum_{j=1}^n y_j = 1,\\ y\ge 0. \end{cases}$$

The pair (**CP**-$\overline{\mathbf{N1}}$) and (**CP**-$\overline{\mathbf{N2}}$) constitute an I-Fuzzy primal-dual pair.

Proof. In (**CP**-$\overline{\textbf{N2}}$), the objective function "$\max\ \chi_1^P - \chi_2^P$" is equivalent to "$\min\ -(\chi_1^P - \chi_2^P)$". Thus, the problem (**CP**-$\overline{\textbf{N2}}$) can be recasted as,

$$(\textbf{CP}-\overline{\textbf{N3}}) \quad \max \quad -(\chi_1^P - \chi_2^P)$$

s.t.

$$\begin{cases} s_0 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \le \beta_0\left(A_i y - v_0\right); & (i=1,2,\ldots,m), \\ t_0 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \ge \beta_0\left(v_0 + s_0 - A_i y\right); & (i=1,2,\ldots,m), \\ 0 \le \chi_2^P \le \chi_1^P \le 1, \\ \chi_1^P + \chi_2^P \le 1, \\ \sum_{j=1}^{n} y_j = 1, \\ y \ge 0. \end{cases}$$

Hence, (**CP**-$\overline{\textbf{N1}}$) and (**CP**-$\overline{\textbf{N3}}$) are a primal-dual pair in I-Fuzzy sense [**13**]. □

4. Numerical Example

To demonstrate the applicability and feasibility of the proposed research, we solve the following numerical example from [**8**] in the following manner:

Consider an $TMGIG$, whose payoff matrix A is

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix}$$

Let u_0 = 2.5, v_0 = 1.5, p_0 = 1, q_0 = 0.6, s_0 = 1, t_0 = 0.75, i.e. $\mathbb{K}1$ aspires to win more than 2.5 but is content if he wins at least 1.5, similarly $\mathbb{K}2$ aspires not to lose more than 1.5 but he will be content if he loose not more than 2.5.

The LPP corresponding to $\mathbb{K}1$ can be defined as follows:

$$(\overline{\textbf{K1}}) \qquad \max \quad \sigma_1^P - \sigma_2^P$$

s.t.

$$\begin{cases} p_0 \ln\left(\sigma_1^P\left(1-e^{-\alpha_0}\right)+e^{-\alpha_0}\right) \le \alpha_0\left(x_1 + 4x_2 - 2.5\right), \\ p_0 \ln\left(\sigma_1^P\left(1-e^{-\alpha_0}\right)+e^{-\alpha_0}\right) \le \alpha_0\left(3x_1 - 2.5\right), \\ q_0 \ln\left(\sigma_2^P\left(1-e^{-\alpha_0}\right)+e^{-\alpha_0}\right) \ge \alpha_0\left(1.5 - x_1 - 4x_2\right), \\ q_0 \ln\left(\sigma_2^P\left(1-e^{-\alpha_0}\right)+e^{-\alpha_0}\right) \ge \alpha_0\left(1.5 - 3x_1\right), \\ 0 \le \sigma_2^P \le \sigma_1^P \le 1, \\ \sigma_1^P + \sigma_2^P \le 1, \\ x_1 + x_2 = 1, \\ x_1, x_2 \ge 0. \end{cases}$$

Similarly, the LPP corresponding to $\mathbb{K}2$ can be defined as follows:

$$(\overline{\mathbf{K2}}) \qquad \max \quad \chi_1^P - \chi_2^P$$
$$\text{s.t.}$$
$$\begin{cases} s_0 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \leq \beta_0\left(1.5-y_1-3y_2\right), \\ s_0 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \leq \beta_0\left(1.5-4y_1\right), \\ t_0 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \geq \beta_0\left(y_1+3y_2-2.5\right), \\ t_0 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \geq \beta_0\left(4y_1-2.5\right), \\ 0 \leq \chi_2^P \leq \chi_1^P \leq 1, \\ \chi_1^P+\chi_2^P \leq 1, \\ y_1+y_2=1, \\ y_1, y_2 \geq 0. \end{cases}$$

By solving the I-Fuzzy LPPs ($\overline{\mathbf{K1}}$) and ($\overline{\mathbf{K2}}$) for various values of the shape parameters α_0 and β_0, we obtain different outcomes for the minimal degree of acceptance (σ_1^P, χ_1^P) and the maximal degree of rejection (σ_2^P, χ_2^P), corresponding to $\mathbb{K}1$ and $\mathbb{K}2$, respectively. The resulting values from these computations are systematically presented in the following Table 1.

Table 1
Optimal Strategies and Acceptance-Rejection Degrees to $\mathbb{K}1$ and $\mathbb{K}2$ for different values of Shape Parameters

α_0	β_0	σ_1^P	σ_2^P	χ_1^P	χ_2^P	$x=(x_1,x_2)$	$y=(y_1,y_2)$	σ_1^P - σ_2^P	χ_1^P - χ_2^P
0.001	0.001	0.49987	0.16659	0.49987	0.33322	(0.66666,0.33333)	(0.50000,0.50000)	0.33327	0.16665
0.010	0.010	0.49875	0.16597	0.49875	0.33222	(0.66666,0.33333)	(0.50000,0.50000)	0.33277	0.16652
0.090	0.090	0.48875	0.16048	0.48875	0.32338	(0.66666,0.33333)	(0.50000,0.50000)	0.32827	0.16536
0.160	0.160	0.48001	0.15575	0.48001	0.31572	(0.66666,0.33333)	(0.50000,0.50000)	0.32425	0.16428
0.280	0.280	0.46505	0.14784	0.46505	0.30275	(0.66666,0.33333)	(0.50000,0.50000)	0.31721	0.16230
0.380	0.380	0.45264	0.14143	0.45264	0.29211	(0.66666,0.33333)	(0.50000,0.50000)	0.31121	0.16053
0.480	0.480	0.44028	0.13519	0.44028	0.28163	(0.66666,0.33333)	(0.50000,0.50000)	0.30509	0.15864
0.580	0.580	0.42802	0.12907	0.42800	0.27134	(0.66666,0.33333)	(0.50000,0.50000)	0.29894	0.15666
0.680	0.680	0.41580	0.12322	0.41580	0.26123	(0.66666,0.33333)	(0.50000,0.50000)	0.29258	0.15457
0.780	0.780	0.40371	0.11750	0.40371	0.25132	(0.66666,0.33333)	(0.50000,0.50000)	0.28621	0.15239
0.880	0.880	0.39174	0.11196	0.39174	0.24161	(0.66666,0.33333)	(0.50000,0.50000)	0.27877	0.15012
0.980	0.980	0.37989	0.10659	0.37989	0.23211	(0.66666,0.33333)	(0.50000,0.50000)	0.27329	0.14778
1.000	1.000	0.37754	0.10554	0.37754	0.23023	(0.66666,0.33333)	(0.50000,0.50000)	0.27199	0.14730
1.500	1.500	0.32082	0.08157	0.32082	0.18245	(0.66666,0.33333)	(0.50000,0.50000)	0.23924	0.13449
2.000	2.000	0.23601	0.03152	0.23601	0.14824	(0.66666,0.33333)	(0.49998,0.50001)	0.20715	0.12077
2.500	2.500	0.22274	0.04620	0.22274	0.11632	(0.66666,0.33333)	(0.49999,0.50000)	0.17654	0.10641
3.000	3.000	0.18242	0.03399	0.18245	0.09001	(0.66666,0.33333)	(0.49999,0.50000)	0.14843	0.09244
3.500	3.500	0.14806	0.02465	0.14805	0.06884	(0.66666,0.33333)	(0.49999,0.50000)	0.12340	0.07921
4.000	4.000	0.11921	0.01768	0.11920	0.05211	(0.66666,0.33333)	(0.49999,0.50000)	0.10152	0.06709
4.500	4.500	0.09535	0.01254	0.09535	0.03911	(0.66666,0.33333)	(0.49999,0.50000)	0.08280	0.05624
5.000	5.000	0.07586	0.00882	0.07585	0.02913	(0.66666,0.33333)	(0.49999,0.49999)	0.06703	0.04672

4.1 *Comparative Analysis*

To compare the dynamics of obtained solutions and efficacy of the developed method a comparitive analysis is performed against existing approaches : Aggarwal et al. [**8**] and Khan et al. [**11**] in the context of $TMGIG$. Based on the results obtained, the following comparison points, distinctively presented with respect to each approach, are shown below.

*(i) Comparison with Aggarwal et al.'s [**8**] Method*

1. The original solution to this numerical problem [**8**] has been calculated by assuming the I-Fuzzy goals with linear membership and non-membership functions. In this paper, the solution to this numerical problem is calculated by taking exponential membership and non-membership functions for I-Fuzzy goals with a calibrated implement: shape parameter.

2. For values of shape parameter α_0 = 0.001 and β_0 = 0.001, the solutions of this numerical problem: σ_1 = 0.49987, σ_2 = 0.16659, χ_1 = 0.49987, χ_2 = 0.33322, $x = (x_1, x_2) = (0.66666, 0.33333)$, $y = (y_1, y_2) = (0.50000, 0.50000)$, coincide with the solution to the problem [**8**].

3. The dynamic nature of the shape parameter is depicted through the vagueness created by it in Table 1. The shape parameter impacts the values of minimal degrees of acceptance and maximal degrees of rejection corresponding to $\mathbb{K}1$ and $\mathbb{K}2$, respectively. As we increase the value of shape parameters α_0 and β_0: 0.001 ≤ α_0 ≤ 5 and 0.001 ≤ β_0 ≤ 5, the values of σ_1^P and σ_2^P corresponding to $\mathbb{K}1$, the values of χ_1^P and χ_2^P corresponding to $\mathbb{K}2$ decrease for the given range of values of α_0 and β_0. Also, the values of $\sigma_1^P - \sigma_2^P$ and $\chi_1^P - \chi_2^P$ decrease gradually for the same values of α_0 and β_0 as discussed, but the constraints 0 ≤ $\sigma_2^P \leq \sigma_1^P$ ≤ 1 and 0 ≤ $\chi_2^P \leq \chi_1^P$ ≤ 1, however small are the obtained values of σ_1^P, σ_2^P, χ_1^P and χ_2^P.

4. The present work, for better understanding and analysis of TZMG, has the possibility to be extended to TZMGs with I-Fuzzy goals and fuzzy payoffs, expressed in the form of exponential membership and non-membership functions.

*(ii) Comparison with Khan et al.'s [**11**] Method*

Prior to provide a comprehensive comparitive analysis, we solved the above-mentioned example by using Khan et al.'s approach [**11**]. Using this approach, the obtained results have been provided in Table 2 for different values. The valuable comparison points have been developed as below:

Table 2
Optimal Solutions for $\mathbb{K}1$ and $\mathbb{K}2$ for different values of Parameter λ

λ	x_1	x_2	ς	y_1	y_2	Φ
0	0.66666	0.33333	0.50000	0.50000	0.50000	0.50000
$\frac{1}{20}$	0.66666	0.33333	0.51666	0.50000	0.50000	0.50000
$\frac{1}{19}$	0.66666	0.33333	0.51753	0.50000	0.50000	0.50876
$\frac{1}{18}$	0.66666	0.33333	0.51850	0.50000	0.50000	0.50925
$\frac{1}{17}$	0.66666	0.33333	0.51960	0.50000	0.50000	0.50980
$\frac{1}{16}$	0.66666	0.33333	0.52083	0.50000	0.50000	0.51042
$\frac{1}{15}$	0.66666	0.33333	0.52220	0.50000	0.50000	0.51110
$\frac{1}{14}$	0.66666	0.33333	0.52380	0.50000	0.50000	0.51190
$\frac{1}{13}$	0.66666	0.33333	0.52563	0.50000	0.50000	0.51281
$\frac{1}{12}$	0.66666	0.33333	0.52776	0.50000	0.50000	0.51388
$\frac{1}{11}$	0.66666	0.33333	0.53030	0.50000	0.50000	0.51515
$\frac{1}{10}$	0.66666	0.33333	0.53333	0.50000	0.50000	0.51666
$\frac{1}{9}$	0.66666	0.33333	0.53703	0.50000	0.50000	0.51851
$\frac{1}{8}$	0.66666	0.33333	0.54166	0.50000	0.50000	0.52083
$\frac{1}{7}$	0.66666	0.33333	0.54763	0.50000	0.50000	0.52381
$\frac{1}{6}$	0.66666	0.33333	0.55553	0.50000	0.50000	0.52776
$\frac{1}{5}$	0.66666	0.33333	0.56666	0.50000	0.50000	0.53333
$\frac{1}{4}$	0.66666	0.33333	0.58333	0.50000	0.50000	0.54166
$\frac{1}{3}$	0.66666	0.33333	0.61110	0.50000	0.50000	0.55555
$\frac{1}{2}$	0.66666	0.33333	0.66666	0.50000	0.50000	0.58333
1	0.66666	0.33333	0.83333	0.50000	0.50000	0.66666

1. The results obtained by Khan et al.'s [11] method in Table 2, it is observed that the optimal strategies to $\mathbb{K}1 : x_1 = 0.66666, x_2 = 0.33333$ and $\mathbb{K}2 : y_1 = 0.50000, y_2 = 0.50000$ remains constant for changing values of the parameter λ, but the satisfaction levels to the optimal strategies of both $\mathbb{K}1$ and $\mathbb{K}2$ increases with increasing values of λ. This is highly noticable from Table 2 specifically when value of λ changes from $\lambda = \frac{1}{2}$ to $\lambda = 1$. Furthermore, this indicates that Khan et al.'s [11] method turns biased towards a particular strategy, neglecting the dynamism inclusive within the goals of $TMGIG$.

2. The parameter λ utilized by Khan et al. [11] provides intricacy in dealing with I-Fuzzy goals within TZMGs by providing detailed geometrical interpretation when computed for numerous values of λ. But it cannot attain the level of sensitivity which the shape parameters (α_0 and β_0) could, even by considering large number of values of λ. This is because the results obtained at a single value of shape parameters (α_0 and β_0) in Table 1 caters sensitivity equivalent to results obtained by computing aggregate results for a $TMGIG$ considering a large number of values of parameter λ.

3. Khan et al.'s [11] method only infers and discusses about the satisfaction level to the strategy of Players $\mathbb{K}1$ and $\mathbb{K}2$ respectively. On the other hand in Table 1 the minimum level of acceptance and maximum degree of rejection is computed simultaneously. This factor plays a crucial role while dealing with some real life problems, which take into account both acceptance and rejection to a strategy.

4. The absence of rejection or dissatisfaction levels to a strategy in Khan et al.'s [11] methodology limits its scope of risk assessment in real-life scenarios, which deal with risk management along

with optimal attainment of results in dealing a problem.

5. MRI Related CIs Adverse Events Reporting Based TZMG

This section is organized into three subsections: the first outlines the adverse effects associated with CI and MRI, the second details the problem formulation and its solution, and the third provides the discussion and key advantages of the proposed research.

5.1 CI and MRI Associated Adverse Effects

Disabilities are a sensitive problem existing within the developing human society. Hearing loss is a significant disability issue which impacts the life of the people socially, psychologically, mentally as well as economically. Hearing loss does not only delimits the physical capabilities of people but it also puts an impact on the mental health creating obstructions in their academic life reducing the scope of their employment. These impacts and obstructions dwindle the confidence of these people as a part of the society. Significant technological advancements in the medical industry have helped in making people's life easier. CIs is one such technology restoring the hearing ability of people, providing pragmatic support, help them through the challenges they had encountered and lead their lives in an inordinately positive manner. CIs are surgically inseminated devices that retrieve external sounds through the medium of an external microphone. The sound is processed and converted into radio-frequency signals magnetically paired over the skin. The receiver decrypts the radio-frequency signals and stimulates the auditory nerve through the electrode array within the cochlea.

Technological advancements in CI machinery have resulted in significant improvements regarding user compatibility. Advancements in the external microphone receiver prominently helps with transient sound processing for sounds of dynamic frequency range along with improved battery life, water resistance and shock absorption. Advancements in the implanted component provides with its higher natural viability for the users. MRI compatibility with CIs is one such advancement that has been worked upon for years by the CI manufacturing companies to achieve ease for patients with CIs to undergo MRIs without facing any internal-external risk or surgical removal for the same. Major worldwide manufacturers such as Advanced Bionics, Cochlear Limited and MED-EL have developed devices which are MRI compatible but as the technology is under-developing users are prone to face some adverse events during MRI. The Food and Drug Administration (FDA), Manufacturer and User Facility Device Experience (MAUDE) of the United States of America runs the initiative to collect information regarding MRI related CI adverse events.

In this section, we utilize the MRI related CI adverse events data by FDA and MAUDE from [**25**], for particular problems associated with the following companies : Advanced Bionics, Cochlear Limited and MED-EL. We use the data to construct an $TMGIG$. Further, we construct a LPP corresponding to the matrix game and solve it using exponential form of functions to find an optimal solution.

5.2 Formulation and Solution of MRI Related CI Adverse Effects as TZMG with I-Fuzzy Goals

Healthcare management system upholds the paramount responsibility to tackle the numerous disabilities, diseases and other significant healthcare issues. Healthcare management system tackles the problem of hearing disability by using various natural therapies, medications, hearing-aids, surgical implants etc. The mentioned methods are utilized based on the percentage of hearing disability with which a person is suffering. CIs is one such revolutionary surgical and technological advancements that helps in resolving very high percentage of hearing disability in peoples. MRI compatibility issues of CIs show up various adverse effects with CI users. These MRI related CI adverse effects pose a significant problem to be resolved by the healthcare management system by providing the best MRI compatible CI solution to the CI users.

This form of interaction between the healthcare management system and the adverse effects associated with MRI incompatibility of CIs indicates that TZMG framework could be utilized to cater the mentioned problem. For 100% reporting of MRI related CI adverse effects by all the CI users, the CIs should be 0% MRI compatible, similarly, 100% MRI compatibility of CIs achieved by manufacturing companies, 0% MRI adverse effect problems should be reported. Therefore the given game can be considered as a TZMG.

For removing the MRI related CI adverse effects problem in a healthcare management system, we consider all major MRI adverse effects created by various CIs in the CI users. The CI users report the following adverse effects : dislocation of the CI, expulsion of the CI, improper procedure, uncodified MRI incompatibility report, experience of pain and failure of CI against the manufacturing companies: Advanced Bionics, Cochlear Limited and MED-EL.

Set the reported adverse effects (MRI incompatibility problems of CI) as $\mathbb{K}1$ and the healthcare management system as $\mathbb{K}2$. In this strategic interaction, $\mathbb{K}1$ represents the various adverse effects encountered by CI users due to its incompatibility with MRI, while $\mathbb{K}2$ represents the healthcare management system which assesses the data of various adverse effects reported by CI users associated with various CI manufacturing companies. In the view of game theory, both players can make strategic decisions and counter each others' moves. Thus, to the following reference adverse effects aim to maximize their gain by presenting maximum recorded reports of MRI incompatibility with CI users. Irreconcilably, healthcare management system seeks to minimize these adverse effect problems reported by CI users encountering MRI incompatibility. For this game model, each player has the following strategies:

Strategies of $\mathbb{K}1$ (Adverse effects)

E_1: Dislocation of the CI;

E_2: Expulsion of the CI;

E_3: Improper procedure;

E_4: Uncodified MRI incompatibility report;

E_5: Experience of pain;

E_6: Failure of CI.

Strategies of $\mathbb{K}2$ (Healthcare management system)

C_1: Prescription of CI of Advanced Bionics as the best suited MRI compatible CI equipment by the healthcare management system;

C_2: Prescription of CI of Cochlear Limited as the best suited MRI compatible CI equipment by the healthcare management system;

C_3: Prescription of CI of MED-EL as the best suited MRI compatible CI equipment by the healthcare management system.

In accordance with the following assumptions, the payoff matrix A to the given problem is expressed as follows:

$$A = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} 40 & 226 & 0 \\ 18 & 205 & 0 \\ 3 & 5 & 3 \\ 51 & 297 & 4 \\ 49 & 158 & 4 \\ 6 & 6 & 8 \end{pmatrix} \end{array}$$

where, a_{ij} = Number of persons who reported experiencing adverse MRI CI effect E_i due to the incompatibility of CI of manufacturing company C_j.

$$\bar{v} = \max_i \max_j (a_{ij}) = \max\{51, 297, 8\} = 297$$

$$\underline{v} = \min_j \min_i (a_{ij}) = \min\{0, 0, 3, 4, 4, 8\} = 0$$

where, the goal $\bar{v}$ and $\underline{v}$ connotes respectively to the maximal and minimal number of MRI incompatible CI's adverse effect problems that can be recorded in the form of strategies E_1, E_2, E_3, E_4, E_5 or E_6 contrary to the healthcare management system which deals them with the strategies C_1, C_2 and C_3.

Let $u_0 = 186$, $p_0 = 74$, $q_0 = 44$ and σ_1^P, σ_2^P be the minimal degree of rejection, maximal degree of rejection associated with $\mathbb{K}1$, then the LPP ($\overline{\mathbf{L1}}$) corresponding to $\mathbb{K}1$ can be defined as:

$$(\overline{\mathbf{L1}}) \qquad \max \quad \sigma_1^P - \sigma_2^P$$

s.t.

$$\begin{cases} 74 \ln\left(\sigma_1^P (1 - e^{-\alpha_0}) + e^{-\alpha_0}\right) \leq \alpha_0(40x_1 + 18x_2 + 3x_3 + 51x_4 + 49x_5 \\ \qquad\qquad + 6x_6 - 186), \\ 74 \ln\left(\sigma_1^P (1 - e^{-\alpha_0}) + e^{-\alpha_0}\right) \leq \alpha_0(226x_1 + 205x_2 + 5x_3 + 297x_4 + 158x_5 \\ \qquad\qquad + 6x_6 - 186), \\ 74 \ln\left(\sigma_1^P (1 - e^{-\alpha_0}) + e^{-\alpha_0}\right) \leq \alpha_0(3x_3 + 4x_4 + 4x_5 + 8x_6 - 186), \\ 44 \ln\left(\sigma_2^P (1 - e^{-\alpha_0}) + e^{-\alpha_0}\right) \geq \alpha_0(112 - 40x_1 - 18x_2 - 3x_3 - 51x_4 \\ \qquad\qquad - 49x_5 - 6x_6), \\ 44 \ln\left(\sigma_2^P (1 - e^{-\alpha_0}) + e^{-\alpha_0}\right) \geq \alpha_0(112 - 226x_1 - 205x_2 - 5x_3 - 297x_4 \\ \qquad\qquad - 158x_5 - 6x_6), \\ 44 \ln\left(\sigma_2^P (1 - e^{-\alpha_0}) + e^{-\alpha_0}\right) \geq \alpha_0 (112 - 3x_3 - 4x_4 - 4x_5 - 8x_6), \\ 0 \leq \sigma_2^P \leq \sigma_1^P \leq 1, \\ \sigma_1^P + \sigma_2^P \leq 1, \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1, \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{cases}$$

Solving the I-Fuzzy LPP ($\overline{\mathbf{L1}}$) for various values of shape parameters α_0, we get different values for minimal degree of acceptance σ_1^P and maximal degree of rejection σ_2^P associated with $\mathbb{K}1$. These calculations are represented in the Table 3 as follows:

Let $v_0 = 111$, $s_0 = 74$, $t_0 = 55$ and χ_1^P, χ_2^P be respectively the minimal degree of rejection, maximal

Table 3
Optimal Strategies and Acceptance-Rejection Degrees of $\mathbb{K}1$ for Different Values of Shape Parameter

α_0	σ_1^P	σ_2^P	$x = (x_1, x_2, x_3, x_4, x_5, x_6)$	σ_1^P - σ_2^P
3.000×10^{-6}	0.75536	0.01936	(0.17995,0.15897,0.11121,0.22917,0.17296,0.14770)	0.73599
3.001×10^{-6}	0.75752	0.01938	(0.18128,0.16022,0.11138,0.22806,0.17242,0.14663)	0.73814
3.002×10^{-6}	0.75526	0.01937	(0.17995,0.15896,0.11121,0.22920,0.17297,0.14768)	0.73589
3.003×10^{-6}	0.75520	0.01937	(0.17995,0.15896,0.11120,0.22921,0.17297,0.14767)	0.73583
3.004×10^{-6}	0.75805	0.01943	(0.18196,0.16099,0.11160,0.22719,0.17204,0.14621)	0.73862
3.005×10^{-6}	0.75801	0.01943	(0.18196,0.16099,0.11159,0.22720,0.17204,0.14619)	0.73857
3.006×10^{-6}	0.75796	0.01943	(0.18196,0.16098,0.11159,0.22721,0.17204,0.14618)	0.73852
3.007×10^{-6}	0.75723	0.01938	(0.18129,0.16020,0.11137,0.22813,0.17244,0.14656)	0.73784
3.008×10^{-6}	0.75718	0.01939	(0.18129,0.16019,0.11136,0.22814,0.17244,0.14654)	0.73779
3.009×10^{-6}	0.75490	0.01937	(0.17995,0.15893,0.11119,0.22929,0.17300,0.14761)	0.73552
3.010×10^{-6}	0.75485	0.01937	(0.17995,0.15893,0.11118,0.22931,0.17300,0.14759)	0.73547
3.011×10^{-6}	0.75772	0.01944	(0.18197,0.16097,0.11158,0.22727,0.17206,0.14612)	0.73828
3.012×10^{-6}	0.75699	0.01939	(0.18129,0.16018,0.11135,0.22819,0.17246,0.14650)	0.73759
3.013×10^{-6}	0.75694	0.01939	(0.18129,0.16018,0.11135,0.22821,0.17246,0.14648)	0.73755
3.014×10^{-6}	0.75464	0.01938	(0.17999,0.15891,0.11117,0.22936,0.17302,0.14755)	0.73526
3.015×10^{-6}	0.75459	0.01938	(0.17995,0.15891,0.11117,0.22937,0.17303,0.14755)	0.73521
3.016×10^{-6}	0.75454	0.01938	(0.17995,0.15890,0.11117,0.22939,0.17303,0.14753)	0.73516
3.017×10^{-6}	0.75449	0.01938	(0.17995,0.15890,0.11117,0.22940,0.17304,0.14751)	0.73511
3.018×10^{-6}	0.75444	0.01938	(0.17995,0.15890,0.11117,0.22941,0.17304,0.14750)	0.73505
3.019×10^{-6}	0.75439	0.01938	(0.17996,0.15889,0.11116,0.22943,0.17305,0.14749)	0.73500
3.020×10^{-6}	0.74134	0.01941	(0.17312,0.15291,0.11064,0.23428,0.17551,0.15351)	0.72192

degree of rejection associated with $\mathbb{K}2$, then the LPP ($\overline{\textbf{L2}}$) corresponding to $\mathbb{K}2$ can be defined as:

$$(\overline{\mathbf{L2}}) \qquad \max \quad \chi_1^P - \chi_2^P$$

s.t.

$$\begin{cases}
74 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \leq \beta_0\left(111-40y_1-226y_2\right), \\
74 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \leq \beta_0\left(111-18y_1-205y_2\right), \\
74 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \leq \beta_0\left(111-3y_1-5y_2-3y_3\right), \\
74 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \leq \beta_0\left(111-51y_1-297y_2-4y_3\right), \\
74 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \leq \beta_0\left(111-49y_1-158y_2-4y_3\right), \\
74 \ln\left(\chi_1^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \leq \beta_0\left(111-6y_1-6y_2-8y_3\right), \\
55 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \geq \beta_0\left(40y_1+226y_2-185\right), \\
55 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \geq \beta_0\left(18y_1+205y_2-185\right), \\
55 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \geq \beta_0\left(3y_1+5y_2+3y_3-185\right), \\
55 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \geq \beta_0\left(51y_1+297y_2+4y_3-185\right), \\
55 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \geq \beta_0\left(49y_1+158y_2+4y_3-185\right), \\
55 \ln\left(\chi_2^P\left(1-e^{-\beta_0}\right)+e^{-\beta_0}\right) \geq \beta_0\left(6y_1+6y_2+8y_3-185\right), \\
0 \leq \chi_2^P \leq \chi_1^P \leq 1, \\
\chi_1^P + \chi_2^P \leq 1, \\
y_1 + y_2 + y_3 = 1, \\
y_1, y_2, y_3 \geq 0.
\end{cases}$$

Solving the I-Fuzzy LPP ($\overline{\textbf{L2}}$) for various values of shape parameters β_0 we get different values for minimal degree of acceptance χ_1^P and maximal degree of rejection χ_2^P associated with $\mathbb{K}2$. These calculations are represented in the Table 4 as follows:

Table 4
Optimal Strategies and Acceptance-Rejection Degrees to $\mathbb{K}2$ for Different Values of Shape Parameters

β_0	χ_1^P	χ_2^P	$y = (y_1, y_2, y_3)$	χ_1^P - χ_2^P
3.000$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.001$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.002$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.003$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.004$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.005$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.006$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.007$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.008$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.009$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.010$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.011$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.012$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.013$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.014$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.015$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.016$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.017$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.017$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.018$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.019$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1
3.020$\times 10^{-6}$	1	0	(0.13250,0.12252,0.74197)	1

5.3 Discussion and Advantages

Social health management is a critical issue for any country. Social health management includes management of issues such as diseases and disabilities. The disabled people can act as a vital human resource, and can play a crucial role in the development of a country. The disability of these people could be dealt with various medicinal, technological and therapeutic solutions. As per [**26**] the World Health Organization (WHO), over 1.5 billion people experience some degree of hearing loss, and around 430 million of them have disabling hearing loss requiring rehabilitation services. This number is projected to rise to 2.5 billion with disabling hearing loss affecting 700 million by 2050. WHO estimates [**27**] that in India there are approximately 63 million people, who are suffering from significant auditory impairment and as per Indian Sign Language Research and Training Centre [**28**], there are 18 million deaf individuals in the country. These people are at a higher risk in the sense of underdevelopment of personality, social anxiety and job opportunities.

To resolve this problem, CIs are used as an exceptional technological advancement. CI is an effective solution restoring hearing abilities in people suffering from significant hearing loss or deafness. The effectiveness and impact of CIs in the life of people, suffering with disabilities is distinctive and revolutionary in comparison to other existing methods (medicinal, surgical, technological and therapeutic) which are used to deal with disabilities. This distinctiveness of CIs lies in the fact that the other methods of resolving the problem of disabilities act as a replacement to the disabled organ or body part in the life of a person with disability. While, CIs act as a revolutionary tool which do not just act as an essential part of the disabled people, but technically revives the ability of the cochlea to send neural signals via auditory nerve to the brain, enabling in them the ability to hear. Hence, we constructed a $TMGIG$, including adverse effects associated with MRI incompatibility of CI: dislocation of the CI, expulsion of the CI, improper procedure, uncodified MRI incompatibility report, experience of pain and failure of CI. These adverse effects act as $\mathbb{K}1$ and our objective was to find the best MRI compatible CI from manufacturing companies: Advanced Bionics, Cochlear Limited and MED-EL. For this, the healthcare management system is considered as $\mathbb{K}2$ which assesses the number of reports recorded, following an adverse effect corresponding to the CI of a particular company. Though, the CIs of all these companies solve the same problem, but we have to find the CI with least number of reports with respect to all the recorded MRI adverse effects problems. Consequently, the healthcare management system chooses the 3^{rd} course of action (CI of MED-EL) as the best strategy to this problem.

6. Conclusion

This paper proposes a solution methodology for the $TMGIG$ problem incorporating exponential fuzzy goals. The approach begins by formulating a framework in which the goals are represented using triangular exponential membership and non-membership functions. Within this framework, the optimization problems for both players are initially transformed into two LPPs characterized by exponential membership and non-membership functions. Furthermore, the duality relationship between these LPPs is established in the context of intuitionistic fuzzy logic.

The effectiveness of the proposed solution procedure is demonstrated through a numerical example. In this example, the significance of shape parameter is tested not only on the optimal strategy values of both players but also on the degrees of minimal acceptance and maximal rejection. A real case study is considered in which a healthcare management system problem to analyze the MRI associated CI adverse effects has been modeled in the form of $TMGIG$. The optimal solution of this problem provides that what CI device will be suitable for the users undergoing MRI with the highest probability. A comparative analysis with a comprehensive discussion of the proposed procedure is given in contrast to the existing methodologies which are used to find the optimal solutions for $TMGIG$.

In future, the authors aim to extend the present research for solving matrix game problems with I-Fuzzy goals and fuzzy payoffs, expressed in the form of exponential membership and non-membership functions.

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Conflict of interest
The authors declare that they have no conflict of interest.

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