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Prioritizing Ventilator Machines in Emergency Scenarios Using Multi-Criteria Decision-Making with Circular Intuitionistic Fuzzy Set and Dombi Operations

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ABSTRACT

In the modern age, the circular intuitionistic fuzzy (IF) set (CIFS) is highly popular because it includes a membership grade and a non-membership grade within the range of the unit interval [0,1], as well as a circular degree within the range of $[0, \sqrt{2}]$. In contrast, a simple IFS is based only on membership and non-membership grades within the range of [0,1]. Moreover, the theory of IFS is unable to handle information in which a circular degree is involved. Hence, the CIFS is the most generalized format of IFS. In decision-making (DM) sciences, multi-criteria decision making (MCDM) is one of the most effective approaches for combining data and opinions from different sources. It evaluates and selects the best option when a decision involves several criteria. The Dombi t-norm (DTNM) and Dombi t-conorm (DTCNM) are crucial tools for aggregating uncertain and imprecise information. The notion of a power aggregation operator (PAO) is also a powerful concept for evaluating the weight vectors (WV) of alternatives by combining them and their attributes. The idea of PAO is considered a valuable tool for the evaluation of WV, providing reliability in aggregated outcomes. Using the concepts of CIFS, PAOs, DTNM, and DTCNM operations, we construct a new family of aggregation operators (AOs) called circular IF Dombi power weighted averaging (CIFDPWA) and circular IF Dombi power weighted geometric (CIFDPWG) operators for finding reliable solutions to MCDM problems. We investigate some fundamental axioms of AOs, such as monotonicity, boundedness, and idempotency. We provide an MCDM algorithm based on the proposed theory. The case study details the best ventilator selection using the developed CIFDPWA and CIFDPWG based on the MCDM approach. We found that GE Healthcare is the best alternative using the CIFDPWA, while Medtronic is the best option using the CIFDPWG operator. To check the applicability of the developed AOs, we made a detailed comparison with some other existing AOs. Finally, it offers some firm conclusions.

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1. Introduction

The ventilator is a critical and essential machine in modern hospitals. It especially plays a key role in the hospital's intensive care unit (ICU). Assessing ventilator machines while considering multiple factors such as ventilation modes, oxygen concentration (FiO_2) control, maintenance and serviceability, and reliability and durability is challenging. Decision-making (DM) tools are invaluable for understanding uncertain and ambiguous information. The MCDM technique helps rank multiple alternatives while considering different attributes, for example, in healthcare, medical diagnosis problems, engineering and design, business and management, and education. These models are helpful in both qualitative and quantitative ways, allowing for more comprehensive data analysis.

In past eras, many mathematicians and data scientists presented various approaches for assessing vague and uncertain information. The concept of fuzzy sets (FS), introduced by [1], is based on membership grade (MG) to represent uncertain details by taking values between 0 and 1. This was considered a major breakthrough in the field of DM sciences. However, researchers later observed that FS still had limitations. The FS could not clearly explain all real-world and theoretical conditions, especially when we are unsure how much we believe or disbelieve in something. To solve this problem, [2] introduced a new idea called intuitionistic FS (IFS), based on MG and non-membership grade (NMG). The roles of FS and IFS are crucial for precisely investigating MCDM problems. With advancements in DM sciences, [3] discussed a new notion called circular IFS (CIFS), which includes a circular grade (CG) along with MG and NMG. This concept can describe fuzzy information with more precision than existing FS and IFS due to the addition of CG.

1.1 Significance of triangular norms

Triangular norm and triangular conorm are symbolized by t-norm (TNM) and t-conorm (TCNM), which play a basic role in fuzzy logic and fuzzy aggregation operator structures, especially in advanced models like CIFS environments. The basic idea of TNM and TCNM was introduced by [4], and many researchers have developed their extensions. For example, the theory of Frank TNM and TCNM was introduced by [5], and the concept of Hamacher TNM and TCNM was proposed by [6]. Aczel–Alsina TNM and TCNM operations were introduced by [7]. Dombi [8] proposed the novel operational laws called Dombi TNM (DTNM) and Dombi TCNM (DTCNM), which gave a new direction to FS theory. The theory of interval-valued Fermatean FS under Dombi operational laws was discussed in [9], and the theory of the MABAC method and Dombi AOs was given in [10]. Confidence level-based AOs under the spherical FS framework were provided by [11]. The concept of Yager AOs for MCDM applications was discussed in [12], and Frank AOs for the solution of MCDM approaches were discussed in [13].

The DTNM and DTCNM operational laws are vital in fuzzy logic, particularly regarding aggregation and fuzzy information fusion. As flexible parametric operators, they allow for the adjustment of the level of intersection (TNM) and union (TCNM) between fuzzy sets (FS) through a control parameter. This flexibility is beneficial in cases where exact DM is required, such as in MCDM, fuzzy control systems, and approximate reasoning. The DTNM and DTCNM are continuous, commutative, associative, and monotonic, which ensures mathematical soundness and reliability in modeling uncertainty and imprecision. Their parameterized nature makes DTNM and DTCNM superior to other operators such as Hamacher, Aczel–Alsina, and Frank TNM and TCNM operations

1.2 Role of power aggregation operators

Aggregation operators (AOs) are useful when working with data, whether the data is incomplete or precise. In many real-life cases, the information we collect can be vague, uncertain, or incomplete, and AOs help combine different fuzzy values into a single value. PAOs are essential because they can represent how different criteria interact by varying the weight of individual inputs using a tunable exponent. This flexibility allows for more nuanced aggregation, accommodating compensatory and non-compensatory behaviors in DM processes. In many situations, there are multiple choices, and options involve some level of uncertainty. Many researchers have proposed different types of AOs for various fuzzy systems and frameworks, such as CIFS-based AOs discussed by [14] and the solution of machine learning problems using CIFS-based AOs provided by [15]. Generalized PAOs and their applications in group DM were discussed by [16]. Linear Diophantine fuzzy PAOs for solving MCDM problems were discussed by [17]. The solution of site selection using Archimedean AOs under the MCDM approach was discussed by [18], and the MCDM algorithm for site selection using q-rung orthopair FS was offered by [19]. The concept of Picture FS for aggregating ambiguous data was given in [20], and reference [21] discussed the Sugeno–Weber operation for assessing MCDM problems. A decision algorithm for investigating MCDM under the Pythagorean FS framework was offered in [22], and the assessment of renewable energy using FS theory was given in [23]. The investigation of physical education using interval-valued Fermatean FS was discussed by [24].

1.3 Overview of case study and MCDM

The ventilator is used to help the patient relax during the breathing process. Doctors use ventilator machines when patients face breathing problems, illnesses, or undergo surgical procedures. Ventilators also remove carbon dioxide and maintain the acid-base balance in cellular functions during diseases. The ventilator is used as a life support system for conditions such as respiratory distress syndrome, pneumonia, chronic obstructive pulmonary disease, and anesthesia during surgeries. Without ventilators in the modern age, treating patients properly in hospitals is impossible. The survival rate would drop significantly without ventilators in hospitals. In emergencies, when patients cannot perform cellular respiration or breathe, the first suggestion is the use of a ventilator machine for patient survival. MCDM is a valuable technique for assessing multiple alternatives when considering many attributes. Many mathematicians and data scientists have proposed solutions for ventilator machine selection using the MCDM model. For example, a decision support system for the assessment of ventilator machines using MCDM was provided by [25], and the mechanical evaluation of ventilator machines using the MCDM approach was discussed by [26]. The investigation of maintenance and management of ventilator machines using the MCDM method was offered by [27]. The solution of MCDM problems using FS theory-based AOs was discussed by [28], and [29] proposed the complex q-rung orthopair FS for the investigation of MCDM problems. The main goal of this article is presented in Figure 1.

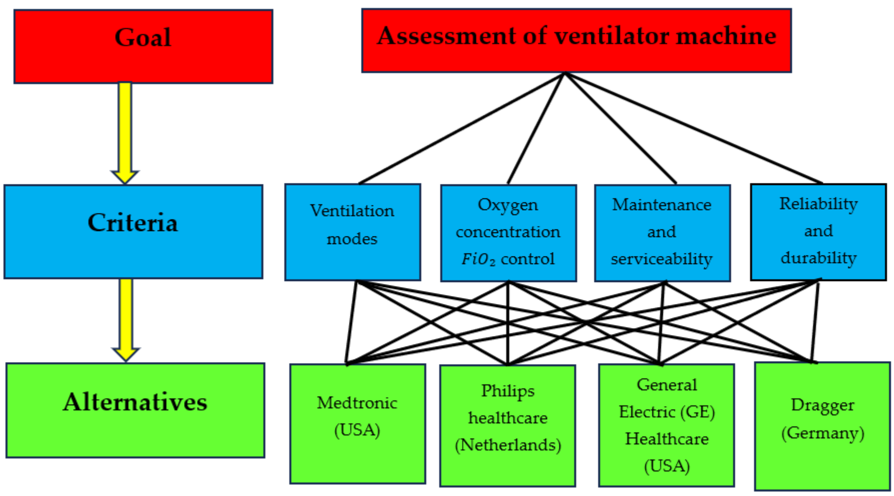


Fig. 1. The main objectives of the proposed theory

The above Figure 1 shows the purpose of the proposed theory. It is noticed that we aim to investigate the best ventilator machine from a list of four world-class ventilator manufacturer companies under consideration of a defined criterion.

1.4 Research gap and motivation

In the past era, there was too much literature given by data scientists for the investigation of MCDM problems under different FS environments. At the beginning of DM sciences, IFS was considered the most precise framework for investigating MCDM problems. The IFS has lost too much information due to the absence of CG in its structure. To overcome this shortcoming, the theory of CIFS is a helpful approach, and it is the generalization of simple IFS. Some silent feature of CIFS is given as follows:

i. When we take NMG and CG are zero, then CIFS is turned into the FS.
ii. When we take CG to be zero, then CIFS is turned into the IFS.
iii. When we take NMG to be zero, then CIFS is turned into the circular FS.

The concept of Dombi operational laws is considered a reliable tool in the data aggregation process due to the presence of flexible parameters. By drawing from the theory of CIFS, Dombi's operational laws, and PAOs, we construct the new family of AOs for precise and accurate investigation of MCDM problems. Some significant key findings are discussed as follows:

i. We discussed the significance of the CIFS framework over the other IFS and FS environments.
ii. Proposed a novel theory of CIFDPWA and CIFDPWG operators.
iii. We have investigated some fundamental axioms of AOs.
iv. Provided a detailed MCDM algorithm.
v. We discussed the case study on investigation of the best world-class ventilator machine manufacturing company under the consideration attributes: ventilation modes, oxygen concentration $FsiO_2$ control, maintenance and serviceability, and reliability and durability.
vi. Provided the solution of the numerical example using the proposed CIFDPWA and CIFDPWG theory.

The graphical view of the research gap is offered in Figure 2.

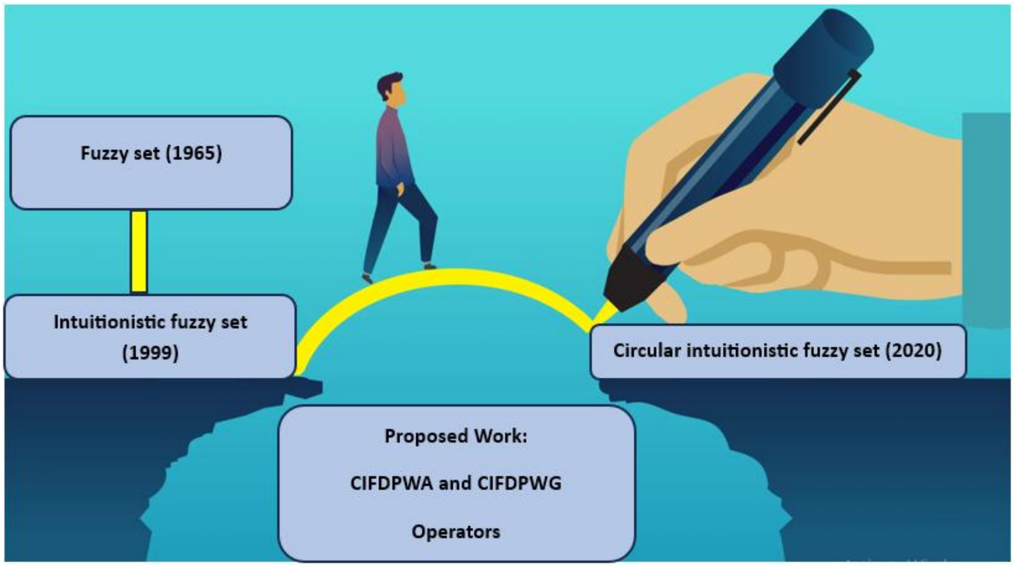


Fig. 2. Depicts of the research gap

The above figure shows that, in the past era, many mathematicians proposed multiple MCDM approaches under the FS and IFS based information. At the same time, our proposed theory is based on the CIFS-based data. CIFS's structure is a more advanced form of FS and IFS due to the addition of CG.

1.5 Organization of the article

The rest of the article is organized as follows: Section 2 contains basic definitions like and operational laws under the CIFS framework. The diagnosed theory of CIFDPWA and CIFDPWG operators is discussed in Section 3, and Section 4 presents the details of the MCDM algorithm. The case study on the investigation of world-class ventilator machine manufacturing companies is discussed in sections 5 and 6, which offers a solution to the numerical problem under the MCDM approach. To validate the proposed theory, we provide a comparative analysis with other existing MCDM models, which is given in section 7, and the conclusion is discussed in section 8. For convenience, the organization of the proposed theory is presented in Figure 3.

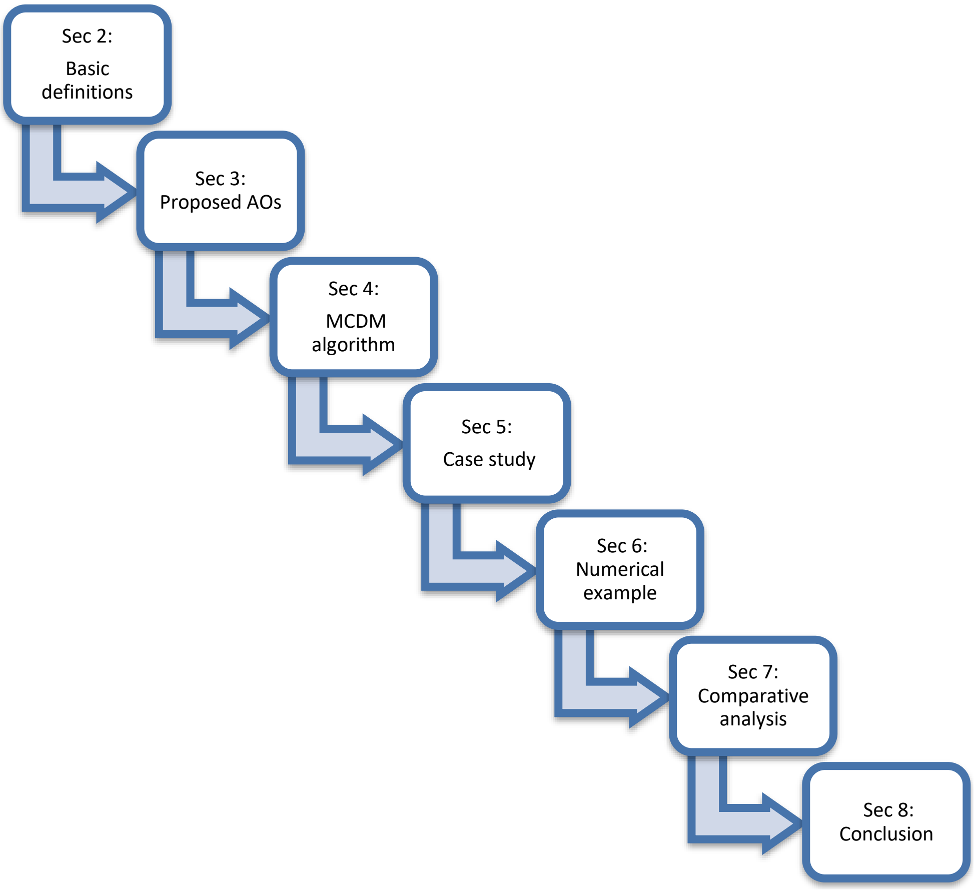


Fig. 3. Graphical view of the organization of the proposed theory

2. Preliminaries

We recall some basic preliminaries in this part to help us understand the diagnostic methodology.

Definition 1. [3] A CIFS ₩ is defined as:

$$₩ = \left\{\left(x, \left(\text{Ꝺ}_{₩}(x), \nu_{₩}(x), \text{ɔ}_{₩}(x)\right)\right) : x \in \bar{\text{T}}\right\}$$

Where $\text{Ꝺ}_{₩}(x): \bar{\text{T}} \rightarrow [0,1]$, $\nu_{₩}(x): \bar{\text{T}} \rightarrow [0,1]$ and $\text{ɔ}_{₩}(x): \bar{\text{T}} \rightarrow [0, \sqrt{2}]$ characterize the MG, NMG, and CG of the element $x \in \bar{\text{T}}$ to the set ₩, and satisfying the condition $0 \leq \text{Ꝺ}_{₩}(x) + \nu_{₩}(x) \leq 1$. Where $(\text{Ꝺ}_{₩}, \nu_{₩}, \text{ɔ}_{₩})$ called the circular IF value (CIFV).

Definition 2. [30] Let $ȍ = (Ꝺ_{ȍ}, v_{ȍ}, ɔ_{ȍ})$ be any CIFV. For for CIFV $ȍ$ score function (SF) is defined as:

$$\emptyset(ȍ) = Ꝺ_{ȍ} - v_{ȍ} - ɔ_{ȍ} \quad (1)$$

Where $\emptyset(ȍ) \in [-1, 1]$.

Th main objective of Equation 1 is to convert multiple aggregated values into a single answer.

Definition 3. [31] Let $ȍ = (Ꝺ_{ȍ}, v_{ȍ}, ɔ_{ȍ})$ be any CIFV. For for CIFV $ȍ$ accuracy function (AF) is defined as:

$$₹(ȍ) = Ꝺ_{ȍ} + v_{ȍ} + ɔ_{ȍ} \quad (2)$$

Where $₹(ȍ) \in [0, 1]$.

Equation 2 is used when two or more two-score values will give the same result, then decision experts (DE) will use the AF formula.

According to the definition $ȍ = (Ꝺ_{ȍ}, v_{ȍ}, ɔ_{ȍ})$ and $ꝑ = (Ꝺ_{ꝑ}, v_{ꝑ}, ɔ_{ꝑ})$ be two CIFVs, then

1. If $\emptyset(ȍ) > \emptyset(ꝑ)$, then $ȍ > ꝑ$
2. If $\emptyset(ȍ) < \emptyset(ꝑ)$ then $ȍ < ꝑ$
3. If $\emptyset(ȍ) = \emptyset(ꝑ) then$

If $₹(ȍ) > ₹(ꝑ)$ then $ȍ > ꝑ$

If $₹(ȍ) < ₹(ꝑ)$ then $ȍ < ꝑ$

If $\emptyset(ȍ) = \emptyset(ꝑ)$ then $ȍ = ꝑ$

Definition 4. [32] Let $ϐ$ and $ḃ$ be two real numbers. Then, DTNM and DTCNM are defined as:

$$Dom(ϐ, ḃ) = \frac{1}{1 + \left\{\left(\frac{1-ϐ}{ϐ}\right)^{ꞙ} + \left(\frac{1-ḃ}{ḃ}\right)^{ꞙ}\right\}^{\frac{1}{ꞙ}}} \quad (3)$$

$$Dom(ϐ, ḃ) = \frac{1}{1 + \left\{\left(\frac{ϐ}{1-ϐ}\right)^{ꞙ} + \left(\frac{ḃ}{1-ḃ}\right)^{ꞙ}\right\}^{\frac{1}{ꞙ}}} \quad (4)$$

Where $m \geq 1$ and $(ϐ, ḃ) \in [0,1] \times [0,1]$.

The concept of DTNM and DTCNM is a valuable tool for aggregating fuzzy and incomplete data. The DTNM and DTCNM provide reliability and authenticity in the aggregation process due to envelopment of flexible parameter $ꞙ$ in their structure.

The theory of PAO is a reliable tool for the precise investigation of the WV of attributes of alternatives.

Definition 5. [33] Let two CIFVs $ȍ = (Ꝺ_{ȍ}, v_{ȍ}, ɔ_{ȍ})$ and $ꝑ = (Ꝺ_{ꝑ}, v_{ꝑ}, ɔ_{ꝑ})$, then the distance between $ȍ$ and $ꝑ$ is calculated as follows:

$$\mathcal{D}(ȍ, ꝑ) = \frac{|Ꝺ_{ȍ} - Ꝺ_{ꝑ}| + |v_{ȍ} - v_{ꝑ}| + |ɔ_{ȍ} - ɔ_{ꝑ}|}{2} \quad (5)$$

Definition 6. [33] Power averaging (PA) can be defined as follows:

$$PA(ϐ_{1,} ϐ_2, \dots, ϐ_n) = \frac{\sum_{i=1}^{n}(1 + ₦(ϐ_i))ϐ_i}{\sum_{i=1}^{n}(1 + ₦(ϐ_i))} \quad (6)$$

Where

$$₦(ϐ_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathcal{S}\ddot{u}\mathcal{P}(ϐ_i, ϐ_j) \quad (7)$$

and $\mathcal{S}\ddot{u}\mathcal{P}(ϐ_i, ϐ_j)$ expresses the support for $ϐ_i$ and $ϐ_j$, which satisfy the conditions:

$$\mathcal{S}\ddot{u}\mathcal{P}(ϐ_i, ϐ_j) \epsilon [0, 1]$$

$$\mathcal{S}\ddot{u}\mathcal{P}(ϐ_i, ϐ_j) = \mathcal{S}\ddot{u}\mathcal{P}(ϐ_i, ϐ_j)$$

$\mathcal{S}\ddot{u}\mathcal{P}(\beta_i,\beta_j) \geq \mathcal{S}\ddot{u}\mathcal{P}(\beta_{\mathcal{K}},\beta_{\mathcal{L}})$, if $|\beta_i-\beta_j| < |\beta_{\mathcal{K}}-\beta_{\mathcal{L}}|$

Definition 7. Let $\ddot{o}=(\text{Ꝺ}_{\ddot{o}},v_{\ddot{o}},\text{ɔ}_{\ddot{o}})$, $\text{ꝑ}=(\text{Ꝺ}_{\text{ꝑ}},v_{\text{ꝑ}},\text{ɔ}_{\ddot{o}})$ be two CIFVs, and $\mathrm{f}\geq 1$ and $\mathfrak{F}>0$ be any real numbers. Then, DTNM and DTCNM for CIFVs are given as follows:

$$1.\ \ddot{o}\oplus\text{ꝑ}=\left(\begin{array}{c}1-\frac{1}{1+\left\{\left(\frac{\text{Ꝺ}_{\ddot{o}}}{1-\text{Ꝺ}_{\ddot{o}}}\right)^{\mathrm{f}}+\left(\frac{\text{Ꝺ}_{\text{ꝑ}}}{1-\text{Ꝺ}_{\text{ꝑ}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}},\\ \frac{1}{1+\left\{\left(\frac{1-v_{\ddot{o}}}{v_{\ddot{o}}}\right)^{\mathrm{f}}+\left(\frac{1-v_{\text{ꝑ}}}{v_{\text{ꝑ}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}},1-\frac{1}{1+\left\{\left(\frac{\text{ɔ}_{\ddot{o}}}{1-\text{ɔ}_{\ddot{o}}}\right)^{\mathrm{f}}+\left(\frac{\text{ɔ}_{\text{ꝑ}}}{1-\text{ɔ}_{\text{ꝑ}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}}\end{array}\right)$$

$$2.\ \ddot{o}\otimes\text{ꝑ}=\left(\begin{array}{c}\frac{1}{1+\left\{\left(\frac{1-\text{Ꝺ}_{\ddot{o}}}{\text{Ꝺ}_{\ddot{o}}}\right)^{\mathrm{f}}+\left(\frac{1-\text{Ꝺ}_{\text{ꝑ}}}{\text{Ꝺ}_{\text{ꝑ}}}\right)^{\mathrm{f}}\right\}},\\ 1-\frac{1}{1+\left\{\left(\frac{v_{\ddot{o}}}{1-v_{\ddot{o}}}\right)^{\mathrm{f}}+\left(\frac{v_{\text{ꝑ}}}{1-v_{\text{ꝑ}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}},\frac{1}{1+\left\{\left(\frac{1-\text{ɔ}_{\ddot{o}}}{\text{ɔ}_{\ddot{o}}}\right)^{\mathrm{f}}+\left(\frac{1-\text{ɔ}_{\text{ꝑ}}}{\text{ɔ}_{\text{ꝑ}}}\right)^{\mathrm{f}}\right\}}\end{array}\right)$$

$$3.\ \mathfrak{F}\ddot{o}=\left(\frac{1}{1+\left\{\mathfrak{F}\left(\frac{1-\text{Ꝺ}_{\ddot{o}}}{\text{Ꝺ}_{\ddot{o}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}},1-\frac{1}{1+\left\{\mathfrak{F}\left(\frac{v_{\ddot{o}}}{1-v_{\ddot{o}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}},\frac{1}{1+\left\{\mathfrak{F}\left(\frac{1-\text{ɔ}_{\ddot{o}}}{\text{ɔ}_{\ddot{o}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}}\right)$$

$$4.\ \ddot{o}^{\mathfrak{F}}=\left(1-\frac{1}{1+\left\{\mathfrak{F}\left(\frac{\text{Ꝺ}_{\ddot{o}}}{1-\text{Ꝺ}_{\ddot{o}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}},\frac{1}{1+\left\{\mathfrak{F}\left(\frac{1-v_{\ddot{o}}}{v_{\ddot{o}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}},1-\frac{1}{1+\left\{\mathfrak{F}\left(\frac{\text{ɔ}_{\ddot{o}}}{1-\text{ɔ}_{\ddot{o}}}\right)^{\mathrm{f}}\right\}^{\frac{1}{\mathrm{f}}}}\right)$$

Theorem 1. If $\ddot{o}=(\text{Ꝺ}_{\ddot{o}},v_{\ddot{o}},\text{Ɔ}_{\ddot{o}})$ and $\text{ꝑ}=(\text{Ꝺ}_{\text{ꝑ}},v_{\text{ꝑ}},\text{Ɔ}_{\text{ꝑ}})$ be CIFV and $\mathfrak{F},\mathfrak{F}_1,\mathfrak{F}_2$ be any three positive real values, then we have

1. $\ddot{o}\oplus\text{ꝑ}=\text{ꝑ}\oplus\ddot{o}$
2. $\ddot{o}\otimes\text{ꝑ}=\text{ꝑ}\otimes\ddot{o}$
3. $\mathfrak{F}(\ddot{o}\oplus\text{ꝑ})=\mathfrak{F}\ddot{o}\oplus\mathfrak{F}\text{ꝑ}$
4. $(\mathfrak{F}_1+\mathfrak{F}_2)\ddot{o}=\mathfrak{F}_1\ddot{o}+\mathfrak{F}_2\ddot{o}$
5. $(\ddot{o}\otimes\text{ꝑ})^{\mathfrak{F}}=\ddot{o}^{\mathfrak{F}}\otimes\text{ꝑ}^{\mathfrak{F}}$
6. $\ddot{o}_1^{\mathfrak{F}}\otimes.\ddot{o}_2^{\mathfrak{F}}=\ddot{o}^{\mathfrak{F}_1+\mathfrak{F}_2}$

3. Diagnosed theory

In this part, we explain the theory of CIFDPWA, CIFDPWG operators using the PAOs, DTNM, and DTCNM operations under the CIFSs information.

Definition 8. Let $\ddot{o}_i=(\text{Ꝺ}_i,v_i,\text{ɔ}_i),(i=1,2,\ldots,n)$ be a collection of CIFVs, and then, their aggregated value using the CIFDPWA operator is also a CIFV, and where $\xi=(\xi_1,\xi_2,\ldots\xi_n)^t$ be the WV of $\ddot{o}_i(i=1,2,\ldots,n)$, such that $0\leq\xi_i\leq 1$for $i=1,2,3,\ldots n$ and $\sum_{i=1}^{n}\xi_i=1$.

$$\text{CIFDPWA}\,(q_1,q_2,\ldots\ldots,q_{n})=q_i\xi_i=\xi_1q_1\oplus\xi_2q_2\oplus\ldots\oplus\xi_{n}q_{n}$$

Theorem 2. Let $\alpha_i=(\text{Ꝺ}_i,v_i,\text{ɔ}_i),(i=1,2,\ldots,n)$ be CIFVs, and then, their aggregated result using the CIFDPWA operation is also a CIFV, and

$$CIFDPWA(q_1, q_2, \dots \dots, q_{ṇ}) = \oplus_{i=1}^{n} \xi_i ȍ_i = \begin{pmatrix} 1 - \dfrac{1}{1 - \left\{\sum_{i=1}^{n} \xi_i \left(\dfrac{ƍ_{ȍ}}{1 - ƍ_{ȍ}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ \dfrac{1}{1 + \left\{\sum_{i=1}^{n} \xi_i \left(\dfrac{v_{ȍ-1}}{v_{ȍ}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{n} \xi_i \left(\dfrac{ɔ_{ȍ}}{1 - ɔ_{ȍ}}\right)^{f}\right\}^{\frac{1}{f}}} \end{pmatrix}$$

Where $\xi = (\xi_{1,} \xi_2, \dots \xi_{,n})^t$ be the WV of $\alpha_i (i = 1,2, \dots, n)$, and $\sum_{i=1}^{n} \xi_i = 1$.

Proof can be seen in the appendix.

Theorem 3. If $\alpha_i = (ƍ_i, v_i, ɔ_i)$ Some CIFVs are identical to other CIFVs, and then, their aggregated value using the CIFDPWA operation is also a CIFV, and $\alpha_i = (ƍ_i, v_i, ɔ_i)$.

CIFDPWA $(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha_i$

Proof can be seen in the appendix.

Theorem 4. If $\alpha_i = (ƍ_{ȍ}, v_{ȍ}, ɔ_{ȍ})$ be some CIFVs and $ȍ^- = min\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\alpha^+ = max\{ȍ_1, ȍ_2, \dots, ȍ_n\}$. Then

$$ȍ^- \leq CIFDWA\ (\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+$$

Proof can be seen in the appendix.

Theorem 5. Consider $\alpha_i = (ƍ_{ȍ}, v_{ȍ}, ɔ_{ȍ}), (i = 1,2, \dots, n)$ are the CIFVs, where $u_i \leq u_i, v_i \geq v_i, ɔ_i \leq ɔ_i$.

$$CIFDPWA(\alpha_1, \alpha_2, \dots, \alpha_n) \leq CIFDPWA(\alpha_1, \alpha_2, \dots, \alpha_n)$$

Where $\xi = (\xi_{1,} \xi_2, \dots \xi_{,n})^t$ be the WV of $\alpha_i (i = 1,2, \dots, n), \xi_{i,} < 0$ and $\sum_{i=1}^{n} \xi_i = 1$.

Definition 9. Consider $ȍ_i = (ƍ_i, v_i, ɔ_i), (i = 1,2, \dots, n)$ be a collection of CIFVs, and then, their aggregated value using the CIFDPWG operator is also a CIFV, and where $\xi = (\xi_{1,} \xi_2, \dots \xi_{,n})^t$ be the of $ȍ_i (i = 1,2, \dots, n)$, such that $0 \leq \xi_i \leq 1$for $i = 1,2,3, \dots n$ and $\sum_{i=1}^{n} \xi_i = 1$.

$$\text{CIFDPWG}\ (q_1, q_2, \dots \dots, q_{ṇ}) = q_i \xi_i = \xi_1 q_1 \oplus \xi_2 q_2 \oplus \dots \oplus \xi_{ṇ} q_{ṇ}$$

Theorem 6. Consider $\alpha_i = (ƍ_i, v_i, ɔ_i), (i = 1,2, \dots, n)$ be CIFVs, and then, their aggregated result using the CIFDPWG operation is also a CIFV, and

$$CIFDPWG(q_1, q_2, \dots \dots, q_{ṇ}) = \otimes_{i=1}^{n} \xi_i ȍ_i = \begin{pmatrix} \dfrac{1}{1 + \left\{\sum_{i=1}^{n} \xi_i \left(\dfrac{ƍ_{ȍ-1}}{ƍ_{ȍ}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ 1 - \dfrac{1}{1 - \left\{\sum_{i=1}^{n} \xi_i \left(\dfrac{v_{ȍ}}{1 - v_{ȍ}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ 1 - \dfrac{1}{1 - \left\{\sum_{i=1}^{n} \xi_i \left(\dfrac{ɔ_{ȍ}}{1 - ɔ_{ȍ}}\right)^{f}\right\}^{\frac{1}{f}}} \end{pmatrix}$$

Where $\xi = (\xi_{1,} \xi_2, \dots \xi_{,n})^t$ be the WV of $\alpha_i (i = 1,2, \dots, n)$, and $\sum_{i=1}^{n} \xi_i = 1$.

Proof of this theorem is similar to Theorem 2. It also satisfies the fundamental axioms of AOs.

4. The MCDM algorithm based on the proposed theory

This section presents the detailed MCDM algorithm based on the diagnosed theory of CIFDPWA and CIFDPWG operators. The details of the algorithm are provided as follows:

Step 1. Collect information on the shape of CIFVs and constrain the decision matrix R_{ij}. Where rows present the numerical values of alternatives, and columns present the numerical values of attributes.

Step 2. Apply the suggested CIFDPWA and CIFDPWG operators on R_{ij} matrix.

$$CIFDPWA(q_1, q_2, \ldots \ldots, q_{n}) = \oplus_{i=1}^{n} \xi_i \ddot{\eth}_i = \begin{pmatrix} 1 - \frac{1}{1 - \left\{\sum_{i=1}^{n} \xi_i \left(\frac{\text{ᘐ}_{\ddot{\eth}}}{1 - \text{ᘐ}_{\ddot{\eth}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ \frac{1}{1 + \left\{\sum_{i=1}^{n} \xi_i \left(\frac{v_{\ddot{\eth}-1}}{v_{\ddot{\eth}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ 1 - \frac{1}{1 + \left\{\sum_{i=1}^{n} \xi_i \left(\frac{\text{ɔ}_{\ddot{\eth}}}{1 - \text{ɔ}_{\ddot{\eth}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}} \end{pmatrix}$$

and

$$CIFDPWG(q_1, q_2, \ldots \ldots, q_{n}) = \otimes_{i=1}^{n} \xi_i \ddot{\eth}_i = \begin{pmatrix} \frac{1}{1 + \left\{\sum_{i=1}^{n} \xi_i \left(\frac{\text{ᘐ}_{\ddot{\eth}-1}}{\text{ᘐ}_{\ddot{\eth}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ 1 - \frac{1}{1 - \left\{\sum_{i=1}^{n} \xi_i \left(\frac{v_{\ddot{\eth}}}{1 - v_{\ddot{\eth}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ 1 - \frac{1}{1 - \left\{\sum_{i=1}^{n} \xi_i \left(\frac{\text{ɔ}_{\ddot{\eth}}}{1 - \text{ɔ}_{\ddot{\eth}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}} \end{pmatrix}$$

Step 3. Apply the score value formula to convert all aggregated outcomes into a single value.

Step 4. Presented the aggregated results in order.

Step 5. End.

For better understanding, the MCDM algorithm is provided in Figure 4.

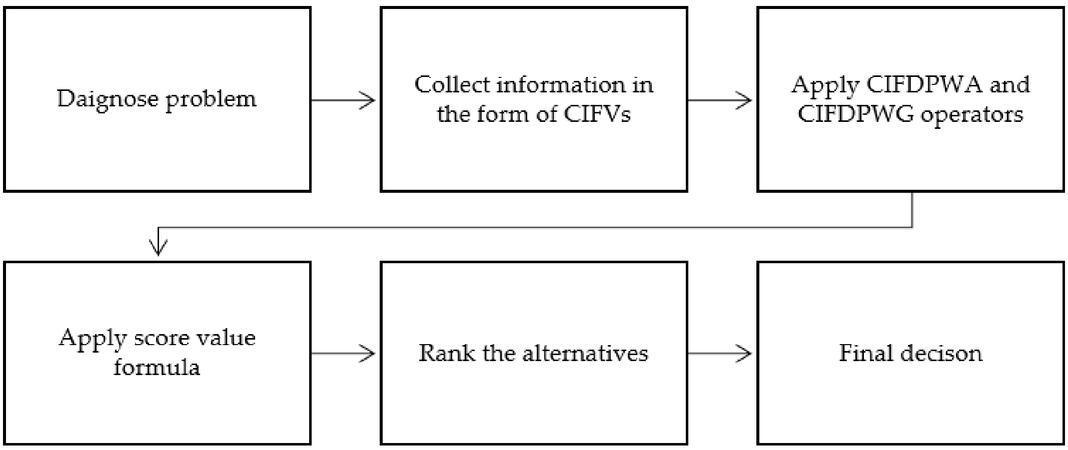


Fig. 4. The proposed MCDM algorithm

The above Figure 4 provides the information about the MCDM algorithm. We noticed that, first of all, we have collected the information based on the IFS system. Then, the proposed CIFDPWA and CIFDPWG theory will be applied. Finally, we rank all the considered alternatives using the score function formula.

5. Case Study

In the current era, medical sciences make progress in leaps and bounds. Several advanced medical equipment that were not present in the past have been invented. In this regard, the ventilator is an advanced device used in the medical field when people have trouble breathing on their own. It gives oxygen to the lungs and removes carbon dioxide from the body. Doctors use ventilators for people who have lung diseases or during surgery. Some ventilators use a mask over the nose or mouth. It is used to keep the person alive during the surgery. Ventilators are used for people with lung diseases like pneumonia, asthma, or COVID-19. It is also used for people who cannot breathe due to accidents, injuries, or drugs, and helps newborns with underdeveloped lungs breathe properly. Before modern machines, the "iron lung," or manually pumped air, was used as a ventilator; ventilators are the most critical medical devices used worldwide during the pandemic. An interesting thing about a ventilator is that it is used not just for hospitals but also by some patients with long-term lung diseases. The graphic interpretation of a ventilator in the medical field is obtainable in Figure 5.

Fig. 5. Graphical view of the ventilator

Many companies produce high-quality ventilators, but the following are recognized for their innovation and reliability. In the modern age, many countries offer the best ventilators. Details of some leading ventilator manufacturers in the world are presented as follows:

i. Medtronic (USA). Medtronic is a big American company that makes medical devices, like pacemakers and insulin pumps. It started in 1949 in a small garage in Minnesota. At first, they repaired hospital equipment. Today, Medtronic is one of the world's biggest medical technology companies, helping millions of heart disease patients by providing the best ventilators.

ii. Philips Healthcare (Netherlands). Philips healthcare is a part of the Dutch company Philips**,** known for its electronics and healthcare innovations. The company was founded in 1891 in the Netherlands, initially making light bulbs. Over time, Philips expanded into healthcare technology in the 20^{th} century**,** developing X-ray machines, medical imaging devices, and patient monitoring systems. Today, Philips Healthcare is a leader in making advanced medical equipment like ventilators.

iii. General Electric (GE) Healthcare (USA). GE Healthcare is an American company that makes medical machines like ventilators. It started in 1892 as part of GE and became a leader in healthcare technology. Today, GE healthcare works in over 100 countries, using artificial intelligence (AI) and digital tools to help doctors diagnose and treat patients better.

iv. Dragger (Germany). Dragger is a German company that started in 1889. It makes hospital machines like ventilators, patient monitors, and safety equipment for firefighters and industries. Dragger's technology helps doctors, hospitals, and emergency workers in over 190 countries around the globe.

The selection of the best ventilator depends upon the attributes: ventilation modes, oxygen concentration FiO_2 control, maintenance and serviceability, and reliability and durability. The details of attributes are given as follows:

i. Ventilation modes. Accessibility and flexibility of ventilation modes are essential for individual and effective respiratory care. Ventilation modes are the different mechanisms a ventilator has to provide breaths for a patient and adjust them to their respiratory requirements. These modes are volume-controlled ventilation (VCV), which provides the preset air volume, and pressure-controlled ventilation (PCV), which includes air until it reaches the set pressure. All modes are aimed at different clinical scenarios, from supportive care of sedated patients to partial support during weaning.

ii. Oxygen concentration FiO_2 control. Oxygen concentration FiO_2 control is the feature of a ventilator to control the fraction of inspired oxygen given to the patient. It is generally between 21% (room air) and 100%, thus rendering correct oxygen therapy based on the patient's status. Frequent monitoring of FiO_2 is vital in handling respiratory distress patients, and the same is required in protecting the patients against oxygen toxicity and providing them with an adequate level of oxygen. A new generation of ventilators provides continual adjustment of FiO_2 to conditions. This aspect permits safe and effective respiratory control, especially for the critical care zones.

iii. Maintenance and serviceability. The maintenance and serviceability of ventilation are critical in providing sustainability and safety of ventilators in the long term. Regular maintenance involves cleaning filters, checking sensors, calibrating systems, and updating software. Ready access to parts and clear maintenance instructions minimizes downtime and maximizes equipment life. Serviceability also relies on the availability of spare parts and the manufacturer's technical support. Properly maintained ventilators would reduce the risk of failure during critical patient care and make the operation cost-effective.

iv. Reliability and durability. Reliability and durability make a significant difference in high-demand or emergency conditions, which are of prime importance regarding consistent performance. A good ventilator should operate precisely, continuously, and not break down regularly. Durability is the quality of a ventilator's endurance to the physical stresses, surroundings, and prolonged usage.

6. Numerical example

This section deals with assessing ventilator manufacturing companies from the list of companies using the MCDM approach. We use the proposed CIFDPWA and CIFDPWG to choose the best ventilator company from the four leading manufacturing companies listed. We select from a list of four different types of ventilators, such as $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4)$: Medtronic, Philips Health Care, GE Healthcare, and Drager, denoted as $\tilde{\alpha}_i = (i = 1, 2, 3, 4)$. Consider we have the following attributes like $ƀ_1$ is the ventilation mode, $ƀ_2$ is oxygen concentration FiO_2 control, $ƀ_3$ is maintenance and serviceability, and $ƀ_4$ is reliability and durability. The weight of each attribute ($ƀ_1$, $ƀ_2$, $ƀ_3$, $ƀ_4$) be $\zeta = (0.25, 0.05, 0.47, 0.07)$ distributed by the decision-makers. Taking the decision maker's opinion, we construct the decision matrix R_{ij} Using CIFVs-based information.

Step 1. Collect information and develop the decision matrix R_{ij} As shown in Table 1.

Table 1
Decision matrix R_{ij}

Alt.	$\mathfrak{b}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_3$	$\mathfrak{b}_4$
$\tilde{\alpha}_1$	$(0.1, 0.3, 0.2)$	$(0.2, 0.3, 0.2)$	$(0.1, 0.2, 0.3)$	$(0.2, 0.5, 0.1)$
$\tilde{\alpha}_2$	$(0.2, 0.2, 0.3)$	$(0.1, 0.4, 0.1)$	$(0.2, 0.1, 0.2)$	$(0.1, 0.3, 0.4)$
$\tilde{\alpha}_3$	$(0.4, 0.1, 0.4)$	$(0.3, 0.2, 0.4)$	$(0.1, 0.1, 0.2)$	$(0.1, 0.1, 0.3)$
$\tilde{\alpha}_4$	$(0.2, 0.4, 0.3)$	$(0.4, 0.2, 0.2)$	$(0.2, 0.3,0.2)$	$(0.3, 0.3, 0.2)$

Step 2. We apply the developed theory of CIFDPWA and CIFDPWG operation on Table 3 above, and the aggregated outcomes are presented in Table 2.

Table 2
Aggregation finding

Alt.	CIFDPWA	CIFDPWG
$\tilde{\alpha}_1$	$(0.0592, 0.3363, 0.1745)$	$(0.1723, 0.1490, 0.3810)$
$\tilde{\alpha}_2$	$(0.1177, 0.1869, 0.1422)$	$(0.3062, 0.0851, 0.3309)$
$\tilde{\alpha}_3$	$(0.1491, 0.1714, 0.1787)$	$(0.1888, 0.0596, 0.3418)$
$\tilde{\alpha}_4$	$(0.1231, 0.4572, 0.1383)$	$(0.3188, 0.2085, 0.3335)$

Step 3. We apply the score function formula discussed in Definition 2 to find the best alternative among all aggregated results. Results are presented in Table 3, given as follows:

Table 3
Scores values

Alt.	CIFDPWA	CIFDPWG
$\tilde{\alpha}_1$	-0.0484	-0.009
$\tilde{\alpha}_2$	-0.0098	-0.0732
$\tilde{\alpha}_3$	-0.0038	0.0441
$\tilde{\alpha}_4$	-0.0462	0.0368

Step 4. For better understanding, we present the ranking results of Table 3 in descending order in Table 4.

Table 4
Ranking order

CIFDWPA	$\tilde{\alpha}_3 > \tilde{\alpha}_1 > \tilde{\alpha}_4 > \tilde{\alpha}_2$
CIFDPWG	$\tilde{\alpha}_1 > \tilde{\alpha}_3 > \tilde{\alpha}_4 > \tilde{\alpha}_2$

For better understanding, we also presented our aggregated outcome in the graph. The graphical view is presented in Figure 6.

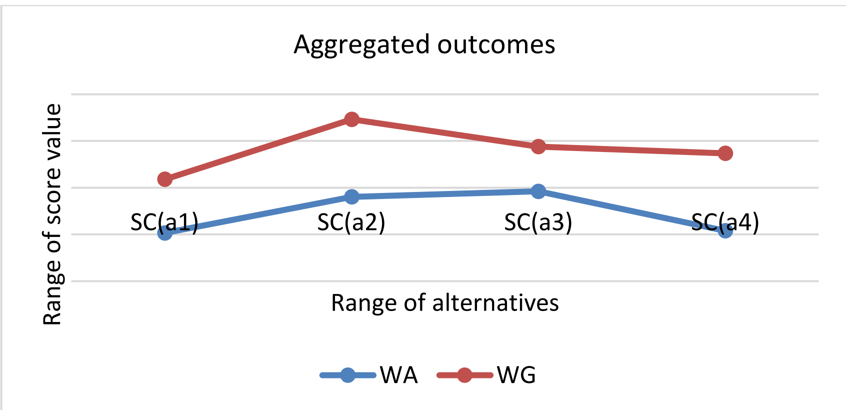


Fig. 6. Geometrical representation of score values

It is noticed that from the above Figure 6, the alternative $\tilde{\alpha}_3$ is the best alternative by using the diagnosed theory of CIFDPWA and $\tilde{\alpha}_3$ be the best option using the developed CIFDPWG theory.

6.1 Practical implementations

In the modern age, decision-making is involved in almost all aspects of life. So, the selection of the best and reliable decision support system for a decision-making problem is a challenging task. It is clearly observed from the above numerical example that our proposed theory of CIFDPWA and CIFDPWG operators is a suitable tool for aggregation of unclear and incomplete information where multiple alternatives and attributes are involved.

For example, DE wants to select the best country of site for a peaceful and joyful tour. In this regard, DE has considered the set of multiple countries like China, Pakistan, the Maldives, and Nepal. Also, they have defined the set attributes such as natural attraction, cultural and historical background, cost and affordability, safety and security. In this practical Sanrio, our proposed theory, CIFDPWA and CIFDPWG operators are the best for finding the most suitable alternative based on the defined attributes.

7. Comparative analysis

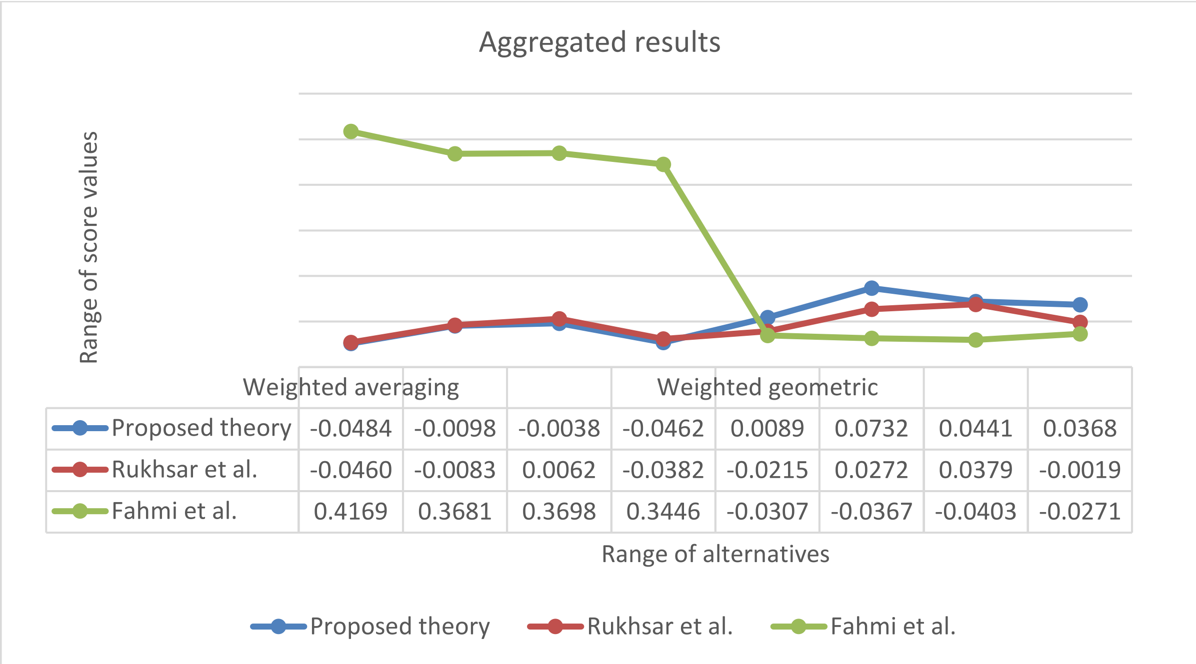
This section presents the comparative analysis of the proposed theory with other existing MCDM models. This section's main objective is to discuss the validation and applicability of the diagnosed theory and the precision in aggregated outcomes. In this regard, we make a deep comparison with existing AOs, such as [34], who developed the circular IF weighted average (CIFWA) and circular IF geometric (CIFWG) operators and circular IF Hamacher (CIFH) weighted averaging (CIFHWA) and CIFH weighted geometric (CIFHWG) theories, presented by [14].

On the other hand, many AOs have failed to investigate the CIFVs-based information due to a lack of sufficient structure (no concept of CG), for example, IF Dombi weighted averaging (IFDWA) and IF Dombi weighted geometric (IFDWG) operators proposed by [35], and IF Dombi power weighted (IFDPW) geometric (IFDPWG) and IFDPW averaging (IFDPWA) operators proposed by [36]. Xia *et al.,* [37] diagnosed the IF Archimedean weighted average (IFAWA) and IF Archimedean weighted geometric (IFAWG) operators. The comparative analysis results with other MCDM models are presented in Table 5.

Table 5
Comparative analysis with existing MCDM models

Method	Operator	Ranking
Proposed work	CIFDPWA	$\tilde{\alpha}_3 > \tilde{\alpha}_1 > \tilde{\alpha}_4 > \tilde{\alpha}_2$
	CIFDPWG	$\tilde{\alpha}_1 > \tilde{\alpha}_3 > \tilde{\alpha}_4 > \tilde{\alpha}_2$
Rukhsar *et al.*, [34]	CIFWA	$\breve{\alpha}_3 > \breve{\alpha}_2 > \breve{\alpha}_4 > \breve{\alpha}_1$
	CIFWG	$\breve{\alpha}_3 > \breve{\alpha}_2 > \breve{\alpha}_4 > \breve{\alpha}_1$
Fahmi *et al.,* [14]	CIFHWA	$\breve{\alpha}_1 > \breve{\alpha}_3 > \breve{\alpha}_2 > \breve{\alpha}_4$
	CIFHWG	$\breve{\alpha}_4 > \breve{\alpha}_1 > \breve{\alpha}_2 > \breve{\alpha}_3$
Seikh and Mandal [35]	IFDWA	No output
	IFDWG	No output
Ameer and Yousaf [36]	IFDPWA	No output
	IFDPWG	No output
Xia *et al.,* [37]	IFAWA	No output
	IFAWG	No output

For a better understanding, we compared our diagnosed theory with other existing CIFWA, CIFWG, CIFHWA, and CIFHWG operators. The graphic interpretation of the aggregated comparative analysis is presented in Figure 7.



	Weighted averaging				Weighted geometric			
Proposed theory	-0.0484	-0.0098	-0.0038	-0.0462	0.0089	0.0732	0.0441	0.0368
Rukhsar et al.	-0.0460	-0.0083	0.0062	-0.0382	-0.0215	0.0272	0.0379	-0.0019
Fahmi et al.	0.4169	0.3681	0.3698	0.3446	-0.0307	-0.0367	-0.0403	-0.0271

Fig. 7. Comparative analysis

From above Figure 7, it is noticed that the alternative $\tilde{\alpha}_3$ and $\tilde{\alpha}_1$ are the best options using the CIFDPWA and CIFDPWG are, respectively. While we compare with another present approach like CIFWA, the alternative $\tilde{\alpha}_3$ be the best option, and when we apply the CIFHWA and CIFHWG theory, then the best options are $\tilde{\alpha}_1$ and $\tilde{\alpha}_4$ respectively.

8. Conclusion

The MCDM is a valuable tool for investigating multiple alternatives while simultaneously considering several attributes. In this regard, CIFS is a powerful tool for assessing uncertain and fuzzy information. Dombi's operational laws and PAOs play a crucial role in the data aggregation of complex data sets. By inspiring the CIFS, PAOs, DTNM, and DTCNM operations concept, we have constructed the family of AOs called CIFDPWA and CIFDPWG operators. Also, we have investigated fundamental axioms of AOs. Then, we offered the MCDM algorithm based on the CIFDPWA and CIFDPWG theory to investigate complicated fuzzy information. We have discussed the case study of world-class ventilator manufacturing companies such as Medtronic (USA), Philips Healthcare (Netherlands), GE Healthcare (USA), and Drager (Germany). We aimed to investigate these companies under the consideration of a list of alternatives like ventilation modes, oxygen concentration FiO_2 control, maintenance and serviceability, and reliability and durability. We also solve the numerical problem using the defined MCDM algorithm. To check the validity of the proposed theory, we conduct a comparative analysis of the developed theory with other existing MCDM methods. There are few limitaions are pesented theory such as: our proposed theory is only suitable for CIFS based data set. It is unable to handle higher-level data sets like Pythagorean, q-rung orthopair, Picture FS, etc. Also, our proposed theory cannot aggregate rough set and soft set-based data.

Appendix

Proof of *Theorem 2*.

Now we will prove this theorem by the method of induction

$$\alpha_1 = \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{\left(\dfrac{\text{Ͽ}_{\text{ᵽ}}}{1 - \text{Ͽ}_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ \dfrac{1}{1 + \left\{\left(\dfrac{1 - \nu_{\text{ᵽ}}}{\nu_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ 1 - \dfrac{1}{1 + \left\{\left(\dfrac{\text{ɔ}_{\text{ᵽ}}}{1 - \text{ɔ}_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}} \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{\left(\dfrac{\text{Ͽ}_{\text{ᵽ}}}{1 - \text{Ͽ}_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ \dfrac{1}{1 + \left\{\left(\dfrac{1 - \nu_{\text{ᵽ}}}{\nu_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ 1 - \dfrac{1}{1 + \left\{\left(\dfrac{\text{ɔ}_{\text{ᵽ}}}{1 - \text{ɔ}_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}} \end{pmatrix}$$

When $n = 2$, based on the Dombi operation on CIFVs, we obtain the result CIFDPWA $(\alpha_1, \alpha_2) =$

$$\xi_1\, \alpha_1 \oplus \xi_2\, \alpha_2 = \begin{pmatrix} 1 - \dfrac{1}{\left\{\xi_1 \left(\frac{\text{Ͽ}_{\text{ȍ}}}{1 - \text{Ͽ}_{\text{ȍ}}}\right)^{\text{ꞙ}} + \xi_2 \left(\frac{\text{Ͽ}_{\text{ᵽ}}}{1 - \text{Ͽ}_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ \dfrac{1}{1 + \left\{\xi_1 \left(\frac{1 - \nu_{\text{ȍ}}}{\nu_{\text{ȍ}}}\right)^{\text{ꞙ}} + \xi_2 \left(\frac{1 - \nu_{\text{ᵽ}}}{\nu_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ 1 - \dfrac{1}{1 + \left\{\xi_1 \left(\frac{\text{ɔ}_{\text{ȍ}}}{1 - \text{ɔ}_{\text{ȍ}}}\right)^{\text{ꞙ}} + \xi_2 \left(\frac{\text{ɔ}_{\text{ᵽ}}}{1 - \text{ɔ}_{\text{ᵽ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{2} \xi_i \left(\dfrac{\text{Ͽ}_{\text{ȍ}}}{1 - \text{Ͽ}_{\text{ȍ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ \dfrac{1}{1 + \left\{\sum_{i=1}^{2} \xi_i \left(\dfrac{1 - \nu_{\text{ȍ}}}{\nu_{\text{ȍ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}}, \\ 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{2} \xi_i \left(\dfrac{\text{ɔ}_{\text{ȍ}}}{1 - \text{ɔ}_{\text{ȍ}}}\right)^{\text{ꞙ}}\right\}^{\frac{1}{\text{ꞙ}}}} \end{pmatrix}$$

Hence, the result is valid for $n = 2$.

Suppose that the given result is valid for $n = x$.

Therefore, we have:

$$CIFDPWA\,(\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n) = \oplus_{i=1}^{n} \xi_i \ddot{o}_i = \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{x} \xi_i \left(\dfrac{Ꝺ_{\ddot{o}}}{1 - Ꝺ_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ \dfrac{1}{1 + \left\{\sum_{i=1}^{x} \xi_i \left(\dfrac{1 - v_{\ddot{o}}}{v_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{2} \xi_i \left(\dfrac{Ↄ_{\ddot{o}}}{1 - Ↄ_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}} \end{pmatrix}$$

Now for $\mathrm{n} = \mathrm{x} + 1$.

$$CIFDPWA\,(\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n) = \oplus_{i=1}^{x+1} \xi_i \ddot{o}_i$$

$$= \begin{pmatrix} 1 - \dfrac{1}{\left\{\sum_{i=1}^{x} \xi_i \left(\dfrac{Ꝺ_{\ddot{o}}}{1 - Ꝺ_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ \dfrac{1}{\left\{\sum_{i=1}^{x} \xi_i \left(\dfrac{1 - v_{\ddot{o}}}{v_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{x} \xi_i \left(\dfrac{Ↄ_{\ddot{o}}}{1 - Ↄ_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}} \end{pmatrix} \oplus \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{\xi_{x+1} \left(\dfrac{Ꝺ_{\ddot{o}}}{1 - Ꝺ_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ \dfrac{1}{1 + \left\{\xi_{x+1} \left(\dfrac{1 - v_{\ddot{o}}}{v_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ 1 - \dfrac{1}{1 + \left\{\xi_{x+1} \left(\dfrac{Ↄ_{\ddot{o}}}{1 - Ↄ_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{\sum_{\mathrm{i}=1}^{\mathrm{x}+1} \xi_{\mathrm{i}} \left(\dfrac{Ꝺ_{\ddot{o}}}{1 - Ꝺ_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ \dfrac{1}{1 + \left\{\sum_{\mathrm{i}=1}^{\mathrm{x}+1} \xi_{\mathrm{i}} \left(\dfrac{1 - v_{\ddot{o}}}{v_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}}, \\ 1 - \dfrac{1}{1 + \left\{\sum_{\mathrm{i}=1}^{\mathrm{x}+1} \xi_{\mathrm{i}} \left(\dfrac{Ↄ_{\ddot{o}}}{1 - Ↄ_{\ddot{o}}}\right)^{f}\right\}^{\frac{1}{f}}} \end{pmatrix}$$

Therefore, the result is valid for $n = x + 1$ if it is valid for $n = 2$, hence, using the induction method, the result is valid for all natural numbers.

Proof of *Theorem 3*.

$$CIFDPWA\,(\ddot{\eth}_1,\ddot{\eth}_2,\dots,\ddot{\eth}_n)=\begin{pmatrix}1-\dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{\text{Ꭾ}_{\ddot{\eth}}}{1-\text{Ꭾ}_{\ddot{\eth}}}\right)^{f}\right\}^{\frac{1}{f}}},\\ \dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{1-v_{\ddot{\eth}}}{v_{\ddot{\eth}}}\right)^{f}\right\}^{\frac{1}{f}}},\\ 1-\dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{\text{ɔ}_{\ddot{\eth}}}{1-\text{ɔ}_{\ddot{\eth}}}\right)^{f}\right\}^{\frac{1}{f}}}\end{pmatrix}$$

$$=\begin{pmatrix}1-\dfrac{1}{1+\left(\dfrac{\text{Ꭾ}_{\ddot{\eth}}}{1-\text{Ꭾ}_{\ddot{\eth}}}\right)\{\sum_{i=1}^{n}\xi_i\}^{\frac{1}{f}}},\\ \dfrac{1}{1+\left(\dfrac{1-v_{\ddot{\eth}}}{v_{\ddot{\eth}}}\right)\{\sum_{i=1}^{n}\xi_i\}^{\frac{1}{f}}},\\ 1-\dfrac{1}{\left(\dfrac{\text{ɔ}_{\ddot{\eth}}}{1-\text{ɔ}_{\ddot{\eth}}}\right)\{\sum_{i=1}^{n}\xi_i\}^{\frac{1}{f}}}\end{pmatrix}$$

$$=\begin{pmatrix}1-\dfrac{1}{1+\left(\dfrac{\text{Ꭾ}_{\ddot{\eth}}}{1-\text{Ꭾ}_{\ddot{\eth}}}\right)\{\sum_{i=1}^{n}\xi_i\}^{\frac{1}{f}}},\\ \dfrac{1}{1+\left(\dfrac{1-v_{\ddot{\eth}}}{v_{\ddot{\eth}}}\right)\{\sum_{i=1}^{n}\xi_i\}^{\frac{1}{f}}},\\ 1-\dfrac{1}{\left(\dfrac{\text{ɔ}_{\ddot{\eth}}}{1-\text{ɔ}_{\ddot{\eth}}}\right)\{\sum_{i=1}^{n}\xi_i\}^{\frac{1}{f}}}\end{pmatrix}$$

$$=(\text{Ꭾ}_{\ddot{\eth}},v_{\ddot{\eth}},\text{ɔ}_{\ddot{\eth}})=\alpha$$

Proof of *Theorem 4*.

Consider $\alpha_i=(\text{Ꭾ}_{\ddot{\eth}},v_{\ddot{\eth}},\text{ɔ}_{\ddot{\eth}})(i=1,2,\dots,n)$ be some CIFVs. Let $\ddot{\eth}^-=\min\{\alpha_1,\alpha_2,\dots,\alpha_n\}=\left(\text{Ꭾ}_{\ddot{\eth}i}{}^-,v_{\ddot{\eth}_i}{}^-\right)$ and $\alpha^+=\max\{\alpha_1,\alpha_2,\dots,\alpha_n\}=\left(\text{Ꭾ}_{\ddot{\eth}}{}^+,v_{\ddot{\eth}}{}^+\right)$, where $\text{Ꭾ}_{\ddot{\eth}_i}{}^-=\min \text{Й}$, $\mu_i{}^+=\max \text{Й}$

$$1-\dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{\text{Ꭾ}_{\ddot{\eth}}{}^-}{1-\text{Ꭾ}_{\ddot{\eth}}{}^-}\right)^{f}\right\}^{\frac{1}{f}}}\le 1-\dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{\text{Ꭾ}_{\ddot{\eth}}}{1-\text{Ꭾ}_{\ddot{\eth}}}\right)^{f}\right\}^{\frac{1}{f}}}\le$$

$$1-\dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{\text{Ꭾ}_{\ddot{\eth}}{}^+}{1-\text{Ꭾ}_{\ddot{\eth}}{}^+}\right)^{f}\right\}^{\frac{1}{f}}},\dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{1-v_{\ddot{\eth}}{}^+}{v_{\ddot{\eth}}{}^+}\right)^{f}\right\}^{\frac{1}{f}}}\le$$

$$\dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{1-v_{\ddot{\eth}}}{v_{\ddot{\eth}}}\right)^{f}\right\}^{\frac{1}{f}}}\le\dfrac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\dfrac{1-v_{\ddot{\eth}}{}^-}{v_{\ddot{\eth}}{}^-}\right)^{f}\right\}^{\frac{1}{f}}}\le$$

$$1-\frac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\frac{1-ɔ_{ð}^{-}}{ɔ_{ð}^{-}}\right)^{f}\right\}^{\frac{1}{f}}}\leq 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\frac{ɔ_{ð}}{1-ɔ_{ð}}\right)^{f}\right\}^{\frac{1}{f}}}\leq 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\xi_i\left(\frac{1-ɔ_i^{+}}{ɔ_i^{+}}\right)^{f}\right\}^{\frac{1}{f}}}$$

Therefore

$ð^{-} \leq Cr$ −IFDWA $(\alpha_i, \alpha_{2,\dots,}\alpha_n) \leq \alpha^{+}$

Acknowledgement

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Conflicts of Interest

The authors declare no conflicts of interest.

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