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The Hyperfuzzy VIKOR and Hyperfuzzy DEMATEL Methods for Multi-Criteria Decision-Making

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ABSTRACT

Handling uncertainty in information systems remains an active area of research, with foundational frameworks such as fuzzy sets, rough sets, intuitionistic fuzzy sets, neutrosophic sets, hyperneutrosophic sets, and plithogenic sets, among others. In particular, Hyperfuzzy Sets and SuperHyperfuzzy Sets offer a powerful means to capture multi-layered uncertainty. At the same time, fuzzy set–based decision-making techniques—such as the Analytic Hierarchy Process (AHP), DEMATEL, VIKOR, TOPSIS, and other MCDM methods—have proven highly effective in practice. In this paper, we introduce novel formulations of Hyperfuzzy VIKOR and Hyperfuzzy DEMATEL that extend the classical Fuzzy VIKOR and Fuzzy DEMATEL methods. We further develop corresponding extensions built upon the SuperHyperfuzzy Set paradigm, laying the groundwork for richer, hierarchically structured decision-making models.

1. Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. All concepts considered in this paper are assumed to be finite, unless stated otherwise.

1.1 *Hyperfuzzy Set and n-SuperHyperFuzzy Set*

A fuzzy set assigns to each element a membership degree in $[0, 1]$, thereby capturing uncertainty through more granular membership levels rather than a strict binary classification [1, 2]. Intuitively, a hyperfuzzy set extends the concept of fuzzy sets into a hierarchical structure, allowing for a more refined and flexible representation of uncertainty. In this sense, a hyperfuzzy set generalizes the tra-

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ditional fuzzy set framework [3–6]. As a further generalization, the concept of a superhyperfuzzy set has also been proposed [7]. These frameworks are useful tools for representing and reasoning about multi-level or hierarchical uncertainty. The formal definition is given below.

Definition 1.1 (Universal Set)**.** *A universal set, denoted by U, is the set that contains all elements under consideration in a particular context. Every set discussed is assumed to be a subset of U.*

Definition 1.2 (Base Set)**.** *A base set S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:*

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.3 (Powerset)**.** *The powerset of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:*

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.4 (n-th Powerset)**.** *(cf.[8, 9])*

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Example 1.5 (Real-Life Application of n-th Powerset)**.** *Let*

$$H = \{\text{milk}, \text{bread}, \text{eggs}\}$$

be the set of basic grocery items. Then:

***First powerset** $P_1(H)$:*

$$P_1(H) = \{\{\text{milk}\}, \{\text{bread}\}, \{\text{eggs}\}, \{\text{milk}, \text{bread}\}, \{\text{milk}, \text{eggs}\}, \{\text{bread}, \text{eggs}\}, \{\text{milk}, \text{bread}, \text{eggs}\}\},$$

which represents every possible non-empty shopping basket for a single trip.

***Second powerset** $P_2(H)$:*

$$P_2(H) = P(P_1(H)),$$

the set of all collections of baskets. Concretely, an element of $P_2(H)$ might be

$$\{\{\text{milk}, \text{bread}\}, \{\text{eggs}\}\},$$

modeling, for example, two distinct shopping trips in one week.

***Third powerset** $P_3(H)$:*

$$P_3(H) = P(P_2(H)),$$

the set of all collections of shopping-trip sequences. For instance,

$$\Big\{ \{\{\text{milk}\}, \{\text{bread}, \text{eggs}\}\}, \ \{\{\text{eggs}\}, \{\text{milk}, \text{bread}\}\} \Big\}$$

could represent two different weekly shopping schedules (one for Week 1 and one for Week 2).

In this way, each increase in the "power" adds one more layer of grouping, moving from single baskets to sequences of baskets to collections of sequences, and so on.

Definition 1.6 (Fuzzy Set)**.** *[1, 10] A fuzzy set τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A fuzzy relation on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a fuzzy relation on τ if*

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \textit{for all } y, z \in Y.$$

Definition 1.7 (Hyperfuzzy Set)**.** *[11–13] Let X be a nonempty set. A mapping*

$$\tilde{\mu} : X \to \tilde{P}([0, 1])$$

is called a hyperfuzzy set *over X, where $\tilde{P}([0, 1])$ denotes the family of all nonempty subsets of the interval $[0, 1]$.*

Example 1.8 (Hyperfuzzy "Pleasant Weather" Set)**.** *Let*

$$X = \{\textit{Rainy}, \ \textit{Cloudy}, \ \textit{Sunny}\}.$$

Define the hyperfuzzy set

$$\tilde{\mu}\colon X \longrightarrow \tilde{P}([0, 1])$$

by

$$\tilde{\mu}(\textit{Rainy}) = \{0.1, \ 0.2, \ 0.3\}, \quad \tilde{\mu}(\textit{Cloudy}) = \{0.4, \ 0.5, \ 0.6\}, \quad \tilde{\mu}(\textit{Sunny}) = \{0.7, \ 0.8, \ 0.9, \ 1.0\}.$$

Here each subset of $[0, 1]$ represents the range of degrees to which a given weather condition is considered "pleasant." For example, "Sunny" days may be rated anywhere from 0.7 up to 1.0 in pleasantness, whereas "Rainy" days only from 0.1 up to 0.3.

Definition 1.9 (n-SuperHyperFuzzy Set)**.** *[6, 14] Let X be a non-empty set, and $n \geq 0$ be an integer. An n-*SuperHyperFuzzy Set *is a mapping:*

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0, 1]),$$

where:

- *$\tilde{\mathcal{P}}_n^*(X)$ denotes the family of all non-empty elements of the n-th PowerSet $\mathcal{P}_n^*(X)$, defined recursively as:*

$$\mathcal{P}_0^*(X) = X, \quad \mathcal{P}_1^*(X) = \mathcal{P}(X), \quad \mathcal{P}_n^*(X) = \mathcal{P}(\mathcal{P}_{n-1}^*(X)), \ \textit{for } n \geq 2,$$

  with $\tilde{\mathcal{P}}_n^*(X) = \mathcal{P}_n^*(X) \setminus \{\emptyset\}$.

- *$\tilde{\mathcal{P}}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.*

Structure:

1. *Each element $A \in \tilde{\mathcal{P}}_n^*(X)$ is a non-empty subset within the n-th PowerSet hierarchy of X.*
2. *The mapping $\tilde{\mu}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n^*(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0, 1]$, representing the degrees of membership associated with the subset A.*

Properties:

- *If* $n = 0$, $\tilde{\mathcal{P}}_0^*(X) = X$, *and the structure reduces to a standard fuzzy set:*
$$\tilde{\mu}_0 : X \to [0, 1].$$
- *For* $n = 1$, $\tilde{\mathcal{P}}_1^*(X) = \tilde{\mathcal{P}}(X)$, *and the structure represents a SuperHyperFuzzy Set:*
$$\tilde{\mu}_1 : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1]).$$
- *For* $n \geq 2$, *the structure recursively extends to higher-order fuzzy relationships:*
$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0, 1]).$$

The n*-SuperHyperFuzzy Set generalizes the concept of fuzzy sets to hierarchical and recursive levels of membership, allowing for higher-order relationships and fuzzy degrees associated with subsets of subsets, and so on, up to the* n*-th PowerSet hierarchy of* X.

Example 1.10 (Smartphone Preference via a 2-SuperHyperFuzzy Set)**.** *Let*
$$X = \{\mathit{Battery},\ \mathit{Camera},\ \mathit{Screen}\}$$
be the set of key smartphone features. Then:

First nonempty powerset $\tilde{\mathcal{P}}_1^*(X)$***:***
$$\tilde{\mathcal{P}}_1^*(X) = \big\{\{\mathit{Battery}\},\ \{\mathit{Camera}\},\ \{\mathit{Screen}\},\ \{\mathit{Battery}, \mathit{Camera}\},$$
$$\{\mathit{Battery}, \mathit{Screen}\},\ \{\mathit{Camera}, \mathit{Screen}\},$$
$$\{\mathit{Battery}, \mathit{Camera}, \mathit{Screen}\}\big\}.$$

Second nonempty powerset $\tilde{\mathcal{P}}_2^*(X)$***:***
$$\tilde{\mathcal{P}}_2^*(X) = \mathcal{P}\big(\tilde{\mathcal{P}}_1^*(X)\big) \setminus \{\emptyset\},$$
whose elements are collections of feature-sets. For example,
$$A = \big\{\{\mathit{Battery}\},\ \{\mathit{Camera}, \mathit{Screen}\}\big\} \ \in\ \tilde{\mathcal{P}}_2^*(X),$$
representing a user grouping "Battery alone" and "Camera with Screen" as two preference clusters.

A 2-SuperHyperFuzzy Set mapping $\tilde{\mu}_2$***:*** *Define*
$$\tilde{\mu}_2 \colon \tilde{\mathcal{P}}_2^*(X) \ \longrightarrow\ \tilde{\mathcal{P}}([0, 1]),$$
so that for instance:
$$\tilde{\mu}_2\big(A\big) = \tilde{\mu}_2\big\{\{\mathit{Battery}\}, \{\mathit{Camera}, \mathit{Screen}\}\big\} = \{0.6,\ 0.8\},$$
meaning the user's satisfaction with this two-cluster grouping lies anywhere between 0.6 and 0.8.

Another element,
$$B = \big\{\{\mathit{Battery}, \mathit{Camera}\},\ \{\mathit{Screen}\}\big\},$$
might be assigned
$$\tilde{\mu}_2(B) = \{0.7,\ 1.0\},$$
indicating higher combined preference for "Battery+Camera" together and "Screen" separately.

Thus, the 2-SuperHyperFuzzy Set $\tilde{\mu}_2$ *captures the user's fuzzy degrees of preference over* bundles of feature-bundles, *generalizing a simple fuzzy evaluation to a hierarchical, multi-layered structure.*

Example 1.11 (Smartphone Preference via a 3-SuperHyperFuzzy Set). *Let*

$$X = \{\mathit{Battery},\ \mathit{Camera},\ \mathit{Screen}\}$$

be the set of key smartphone features. Then:

Third nonempty powerset $\tilde{\mathcal{P}}_3^*(X)$**:**

$$\tilde{\mathcal{P}}_3^*(X) = \mathcal{P}\big(\tilde{\mathcal{P}}_2^*(X)\big) \setminus \{\emptyset\},$$

whose elements are collections of 2-level clusters. For instance, define

$$A_1 = \big\{\{\mathit{Battery}\},\ \{\mathit{Camera}, \mathit{Screen}\}\big\} \in \tilde{\mathcal{P}}_2^*(X), \qquad A_2 = \big\{\{\mathit{Battery}, \mathit{Camera}\},\ \{\mathit{Screen}\}\big\} \in \tilde{\mathcal{P}}_2^*(X).$$

Then

$$A = \{\, A_1,\ A_2\} \ \in\ \tilde{\mathcal{P}}_3^*(X),$$

representing two distinct "bundles of feature-bundles."

A 3-SuperHyperFuzzy Set mapping $\tilde{\mu}_3$**:** *Define*

$$\tilde{\mu}_3 \colon \tilde{\mathcal{P}}_3^*(X) \longrightarrow \tilde{\mathcal{P}}([0,1]),$$

so that for example:

$$\tilde{\mu}_3(A) = \{0.4,\ 0.7\},$$

meaning the user's overall satisfaction with this two-bundle grouping lies between 0.4 and 0.7.

Another element of $\tilde{\mathcal{P}}_3^*(X)$ *is*

$$B_1 = \big\{\{\mathit{Battery}\},\ \{\mathit{Screen}\}\big\}, \quad B_2 = \big\{\{\mathit{Camera}\},\ \{\mathit{Battery}, \mathit{Screen}\}\big\},$$

$$B = \{\, B_1,\ B_2\} \in \tilde{\mathcal{P}}_3^*(X),$$

which might be assigned

$$\tilde{\mu}_3(B) = \{0.6,\ 1.0\},$$

indicating a higher preference range for this alternative.

Thus, the 3-SuperHyperFuzzy Set $\tilde{\mu}_3$ *captures fuzzy degrees of preference over* collections of collections of feature-bundles, *extending the hierarchical evaluation to three levels.*

1.2 Fuzzy VIKOR and Fuzzy DEMATEL

In this paper, we present the definitions of Fuzzy VIKOR [15–18] and Fuzzy DEMATEL [19–23] as decision-making methodologies capable of handling uncertainty.

Several related concepts have also been widely studied in the literature, including:

- Hyper AHP [24] and Hyper TOPSIS [24],
- SuperHyper AHP and SuperHyper TOPSIS [24],
- Neutrosophic VIKOR[25, 26],
- Neutrosophic DEMATEL[27–29],
- Neutrosophic AHP [30, 31],

- Fuzzy WASPAS [32, 33],
- Fuzzy ANP [34, 35],
- Fuzzy LINMAP [36–39],
- Intuitionistic Fuzzy AHP [40, 41],
- Fuzzy MACBETH [42, 43],
- Fuzzy ARAS [44, 45], and
- Neutrosophic TOPSIS [46, 47].

These frameworks are commonly applied in multi-criteria decision-making and continue to be areas of active research.

Definition 1.12 (Fuzzy VIKOR Method)**.** *[15, 16] Consider a set of J alternatives*

$$A = \{A_1, A_2, \ldots, A_J\},$$

which are evaluated over n criteria. Suppose that the performance of alternative A_j on criterion i is given by a triangular fuzzy number

$$\tilde{f}_{ij} = (l_{ij}, m_{ij}, r_{ij}), \quad i = 1, \ldots, n, \; j = 1, \ldots, J.$$

Let I_b and I_c denote the indices of benefit (to be maximized) and cost (to be minimized) criteria, respectively. Also, let w_i be the weight of criterion i, with $\sum_{i=1}^{n} w_i = 1$. The fuzzy VIKOR method comprises the following steps:

(i) Determine the Ideal and Nadir Fuzzy Values. *For each criterion i define:*

$$\tilde{f}_i^* = \begin{cases} \max\limits_{1\le j\le J} \tilde{f}_{ij}, & i \in I_b, \\ \min\limits_{1\le j\le J} \tilde{f}_{ij}, & i \in I_c, \end{cases} \qquad \tilde{f}_i^- = \begin{cases} \min\limits_{1\le j\le J} \tilde{f}_{ij}, & i \in I_b, \\ \max\limits_{1\le j\le J} \tilde{f}_{ij}, & i \in I_c. \end{cases}$$

(Here the comparison of fuzzy numbers is understood to be performed by using an appropriate ranking or defuzzification method.)

(ii) Compute the Normalized Fuzzy Difference. *For each alternative A_j and criterion i, define*

$$\tilde{d}_{ij} = \begin{cases} \dfrac{\tilde{f}_i^* - \tilde{f}_{ij}}{\tilde{f}_i^* - \tilde{f}_i^-}, & i \in I_b, \\ \dfrac{\tilde{f}_{ij} - \tilde{f}_i^*}{\tilde{f}_i^- - \tilde{f}_i^*}, & i \in I_c. \end{cases}$$

In actual computations the fuzzy differences may be aggregated via fuzzy arithmetic and then defuzzified to yield the crisp values d_{ij}.

(iii) Compute the Group Utility and Individual Regret Measures. *For each alternative, define the aggregate (weighted) deviation*

$$S_j = \sum_{i=1}^{n} w_i \, d_{ij},$$

and the maximum weighted regret

$$R_j = \max_{1\le i\le n}\{w_i\, d_{ij}\}, \qquad j = 1, \dots, J.$$

(iv) Compute the Compromise Measure. *Define*

$$S^* = \min_j S_j, \quad S^- = \max_j S_j, \quad R^* = \min_j R_j, \quad R^- = \max_j R_j.$$

Then, for a parameter $v \in [0, 1]$ (representing the weight of the group utility strategy) compute for each alternative

$$Q_j = v\,\frac{S_j - S^*}{S^- - S^*} + (1 - v)\,\frac{R_j - R^*}{R^- - R^*}, \quad j = 1, \dots, J.$$

(v) Ranking and Compromise Solution. *The alternatives are ranked in ascending order of Q_j. The alternative(s) with the smallest Q_j value is (are) considered as the compromise solution(s). In addition, acceptability conditions (e.g., an acceptable advantage and stability) may be applied to confirm the final decision.*

Example 1.13. *Consider three alternatives A_1, A_2, and A_3 evaluated on two criteria. Assume that criterion 1 is a benefit criterion and criterion 2 is a cost criterion. The performance values are provided as triangular fuzzy numbers as follows:*

Alternative	Criterion 1 (Benefit)	Criterion 2 (Cost)
A_1	(3, 5, 7)	(6, 8, 10)
A_2	(4, 6, 8)	(5, 7, 9)
A_3	(2, 4, 6)	(7, 9, 11)

Assume the criterion weights are $w_1 = 0.6$ and $w_2 = 0.4$, and let $v = 0.5$. For simplicity, we defuzzify the fuzzy numbers by taking their median values (i.e., the second element in each triplet).

Step 1: Ideal and Nadir Values.

- Criterion 1 (Benefit): *The medians are:* $A_1 : 5$, $A_2 : 6$, $A_3 : 4$. *Thus, the ideal value is*

  $$f_1^* = 6 \quad \text{(from } A_2\text{)},$$

  and the nadir is

  $$f_1^- = 4 \quad \text{(from } A_3\text{)}.$$

- Criterion 2 (Cost): *The medians are:* $A_1 : 8$, $A_2 : 7$, $A_3 : 9$. *Since cost criteria are to be minimized, the ideal value is*

  $$f_2^* = 7 \quad \text{(from } A_2\text{)},$$

  and the nadir is

  $$f_2^- = 9 \quad \text{(from } A_3\text{)}.$$

Step 2: Normalized Differences. *Using the median values, we define the normalized (crisp) differences:*

$$\text{For Criterion 1 (Benefit):} \quad d_{1j} = \frac{f_1^* - f_{1j}}{f_1^* - f_1^-} = \frac{6 - f_{1j}}{6 - 4} = \frac{6 - f_{1j}}{2},$$
$$\text{For Criterion 2 (Cost):} \quad d_{2j} = \frac{f_{2j} - f_2^*}{f_2^- - f_2^*} = \frac{f_{2j} - 7}{9 - 7} = \frac{f_{2j} - 7}{2}.$$

Thus,

- *For A_1: with $f_{11} = 5$ and $f_{21} = 8$, we obtain*

$$d_{11} = \frac{6-5}{2} = 0.5, \quad d_{21} = \frac{8-7}{2} = 0.5.$$

- *For A_2: with $f_{12} = 6$ and $f_{22} = 7$, we obtain*

$$d_{12} = \frac{6-6}{2} = 0, \quad d_{22} = \frac{7-7}{2} = 0.$$

- *For A_3: with $f_{13} = 4$ and $f_{23} = 9$, we obtain*

$$d_{13} = \frac{6-4}{2} = 1, \quad d_{23} = \frac{9-7}{2} = 1.$$

Step 3: Compute S_j and R_j. *Using the weights $w_1 = 0.6$ and $w_2 = 0.4$:*

$$S_j = w_1\, d_{1j} + w_2\, d_{2j}, \qquad R_j = \max\{w_1\, d_{1j},\ w_2\, d_{2j}\}.$$

Thus,

- A_1:

$$S_1 = 0.6\times0.5+0.4\times0.5 = 0.3+0.2 = 0.5, \quad R_1 = \max\{0.6\times0.5,\ 0.4\times0.5\} = \max\{0.3, 0.2\} = 0.3.$$

- A_2:

$$S_2 = 0.6 \times 0 + 0.4 \times 0 = 0, \quad R_2 = \max\{0, 0\} = 0.$$

- A_3:

$$S_3 = 0.6\times1+0.4\times1 = 0.6+0.4 = 1.0, \quad R_3 = \max\{0.6\times1,\ 0.4\times1\} = \max\{0.6, 0.4\} = 0.6.$$

Set

$$S^* = \min\{S_1, S_2, S_3\} = 0, \quad S^- = \max\{S_1, S_2, S_3\} = 1,$$
$$R^* = \min\{R_1, R_2, R_3\} = 0, \quad R^- = \max\{R_1, R_2, R_3\} = 0.6.$$

Step 4: Compute the Compromise Measure Q_j. *With $v = 0.5$, the compromise measure is given by*

$$Q_j = 0.5\, \frac{S_j - S^*}{S^- - S^*} + 0.5\, \frac{R_j - R^*}{R^- - R^*}.$$

Thus, we obtain:

- *For A_1:*

$$Q_1 = 0.5\left(\frac{0.5-0}{1}\right) + 0.5\left(\frac{0.3-0}{0.6}\right) = 0.5(0.5) + 0.5(0.5) = 0.25 + 0.25 = 0.50.$$

- *For A_2:*

$$Q_2 = 0.5\left(\frac{0-0}{1}\right) + 0.5\left(\frac{0-0}{0.6}\right) = 0.$$

- *For A_3:*

$$Q_3 = 0.5\left(\frac{1.0-0}{1}\right) + 0.5\left(\frac{0.6-0}{0.6}\right) = 0.5(1) + 0.5(1) = 0.5 + 0.5 = 1.0.$$

Step 5: Ranking. *Since the alternatives are ranked in increasing order of Q_j, we have*

$$Q_2 = 0, \quad Q_1 = 0.50, \quad Q_3 = 1.0.$$

Thus, the ranking is:

$$A_2 \prec A_1 \prec A_3.$$

In this example the best alternative is A_2, followed by A_1, and then A_3.

Note: *In a full fuzzy VIKOR analysis, all operations on fuzzy numbers (including the subtraction and division in Step (ii) as well as the defuzzification process) are performed using appropriate fuzzy arithmetic methods. Here, for clarity, we have shown the steps using the median values.*

Definition 1.14 (Fuzzy DEMATEL Method). *[19–23] Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of factors. In the fuzzy DEMATEL method, experts assess the direct influence of factor s_i on factor s_j using linguistic terms (e.g., No, Very Low, Low, High, Very High). These linguistic assessments are converted into triangular fuzzy numbers. A triangular fuzzy number $\tilde{a}$ is denoted by*

$$\tilde{a} = (l, m, r),$$

with membership function

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & x < l, \\ \frac{x-l}{m-l}, & l \le x \le m, \\ \frac{r-x}{r-m}, & m \le x \le r, \\ 0, & x > r. \end{cases}$$

A common fuzzy linguistic scale is given by:

Linguistic Term	Triangular Fuzzy Number
No influence	$(0, 0, 0.25)$
Very low influence	$(0, 0.25, 0.5)$
Low influence	$(0.25, 0.5, 0.75)$
High influence	$(0.5, 0.75, 1.0)$
Very high influence	$(0.75, 1.0, 1.0)$

For each ordered pair (s_i, s_j), denote the fuzzy assessment by $\tilde{z}_{ij}$. Forming the fuzzy direct-relation matrix

$$\tilde{Z} = [\tilde{z}_{ij}]_{n\times n},$$

a fuzzy aggregation procedure (for example, using the Converting Fuzzy data into Crisp Scores (CFCS) method) is applied and the fuzzy numbers are defuzzified to yield a crisp matrix

$$Z = [z_{ij}].$$

The next step is to normalize Z. Define the normalization constant

$$s = \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} z_{ij} \right\},$$

and obtain the normalized direct-relation matrix

$$X = \left[x_{ij}\right] \quad \textit{with} \quad x_{ij} = \frac{z_{ij}}{s}, \quad 0 \le x_{ij} \le 1.$$

The total-relation matrix is then calculated as

$$T = X(I - X)^{-1},$$

where I is the $n \times n$ identity matrix.

Define for each factor s_i the following indices:

$$D_i = \sum_{j=1}^{n} t_{ij} \quad \textit{and} \quad R_i = \sum_{j=1}^{n} t_{ji},$$

where D_i is the sum of the ith row of T (representing the total influence given by s_i) and R_i is the sum of the ith column of T (representing the total influence received by s_i). The prominence *of factor s_i is measured by*

$$D_i + R_i,$$

and its net causal effect by

$$D_i - R_i.$$

A positive value of $D_i - R_i$ indicates that s_i is primarily a causal (driving) factor, while a negative value suggests that s_i is primarily an effect (influenced) factor.

Thus, the fuzzy DEMATEL method integrates fuzzy set theory to handle the inherent imprecision in linguistic assessments and the DEMATEL technique to reveal the causal structure among the factors.

Example 1.15. *Consider a system with three factors, i.e., $S = \{s_1, s_2, s_3\}$. Experts evaluate the pairwise influences using the linguistic scale defined above. Suppose that after fuzzy aggregation and defuzzification (using, for instance, the CFCS method), the crisp direct-relation matrix is obtained as*

$$Z = \begin{bmatrix} 0 & 0.5 & 0.25 \\ 0.75 & 0 & 0.5 \\ 0.5 & 0.25 & 0 \end{bmatrix}.$$

First, determine the normalization constant:

$$s = \max\{0 + 0.5 + 0.25,\ 0.75 + 0 + 0.5,\ 0.5 + 0.25 + 0\} = \max\{0.75,\ 1.25,\ 0.75\} = 1.25.$$

Thus, the normalized direct-relation matrix is

$$X = \frac{1}{1.25} Z = \begin{bmatrix} 0 & 0.4 & 0.2 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0.2 & 0 \end{bmatrix}.$$

Next, compute the total-relation matrix by

$$T = X(I - X)^{-1}.$$

While the explicit entries of T can be obtained by inverting $(I - X)$, the essential idea is that for each factor s_i one obtains

$$D_i = \sum_{j=1}^{3} t_{ij} \quad \text{and} \quad R_i = \sum_{j=1}^{3} t_{ji}.$$

For instance, if we hypothetically obtain

$$D_1 = 1.2, \quad D_2 = 1.5, \quad D_3 = 1.3,$$
$$R_1 = 1.1, \quad R_2 = 1.4, \quad R_3 = 1.5,$$

then the net causal effects are

$$D - R = \begin{bmatrix} 1.2 - 1.1 \\ 1.5 - 1.4 \\ 1.3 - 1.5 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.2 \end{bmatrix}.$$

This result indicates that factors s_1 and s_2 are net cause factors (driving factors), whereas s_3 is a net effect factor. A causal diagram can then be constructed by plotting the prominence ($D + R$) on the horizontal axis and the net effect ($D - R$) on the vertical axis.

2. Results

The results of this paper are presented as follows.

2.1 Generalized VIKOR Methods

The definition of the Generalized VIKOR Methods, generalized from the perspective of Hyperfuzzy and Superhyperfuzzy approaches, is presented below.

Definition 2.1 (Hyperfuzzy VIKOR Method)**.** *Let $A = \{A_1, A_2, \dots, A_J\}$ be a set of alternatives evaluated on n criteria. Suppose the performance of alternative A_j on criterion i is given by a* hyperfuzzy number

$$\hat{f}_{ij} \in \mathcal{HF}([0,1]) = \left\{ \tilde{F} \subseteq [0,1] \mid \tilde{F} \neq \emptyset \right\}.$$

Assume that the set of criteria is partitioned into benefit criteria I_b and cost criteria I_c, and that each criterion has a weight w_i, with $\sum_{i=1}^{n} w_i = 1$. The Hyperfuzzy VIKOR method is defined by the following steps:

1. ***Ideal and Nadir Hyperfuzzy Values.*** *For each criterion i, define*

$$\hat{f}_i^* = \begin{cases} \sup_{1 \le j \le J} \hat{f}_{ij}, & i \in I_b, \\ \inf_{1 \le j \le J} \hat{f}_{ij}, & i \in I_c, \end{cases} \qquad \hat{f}_i^- = \begin{cases} \inf_{1 \le j \le J} \hat{f}_{ij}, & i \in I_b, \\ \sup_{1 \le j \le J} \hat{f}_{ij}, & i \in I_c. \end{cases}$$

   (Here, the supremum and infimum are taken in the order of the hyperfuzzy set.)

2. ***Normalized Hyperfuzzy Difference.*** *For each alternative A_j and criterion i, define*

$$\hat{d}_{ij} = \begin{cases} \dfrac{\hat{f}_i^* \ominus \hat{f}_{ij}}{\hat{f}_i^* \ominus \hat{f}_i^-}, & i \in I_b, \\[2mm] \dfrac{\hat{f}_{ij} \ominus \hat{f}_i^*}{\hat{f}_i^- \ominus \hat{f}_i^*}, & i \in I_c, \end{cases}$$

where $\ominus$ (and the corresponding division) denotes the subtraction and scalar division operations defined in the hyperfuzzy arithmetic framework.

3. ***Group Utility and Individual Regret Measures.*** *Define the hyperfuzzy group utility for each alternative by*
$$\hat{S}_j = \bigoplus_{i=1}^{n} w_i \odot \hat{d}_{ij},$$
and the hyperfuzzy regret measure by
$$\hat{R}_j = \operatorname*{MAX}_{1 \le i \le n}\{w_i \odot \hat{d}_{ij}\},$$
where $\odot$ denotes scalar multiplication in the hyperfuzzy domain, $\bigoplus$ denotes the hyperfuzzy sum, and MAX is the hyperfuzzy maximum.

4. ***Compromise Measure.*** *Let*
$$\hat{S}^* = \min_j \hat{S}_j, \quad \hat{S}^- = \max_j \hat{S}_j, \quad \hat{R}^* = \min_j \hat{R}_j, \quad \hat{R}^- = \max_j \hat{R}_j.$$
Then, for a parameter $v \in [0,1]$, define the compromise measure as
$$\hat{Q}_j = v\,\frac{\hat{S}_j \ominus \hat{S}^*}{\hat{S}^- \ominus \hat{S}^*} \oplus (1-v)\,\frac{\hat{R}_j \ominus \hat{R}^*}{\hat{R}^- \ominus \hat{R}^*},$$
where $\oplus$ denotes the hyperfuzzy addition.

5. ***Ranking.*** *Finally, the alternatives are ranked by the ordering induced by the hyperdefuzzified values*
$$Q_j = \mathcal{D}_{\mathcal{HF}}(\hat{Q}_j),$$
where $\mathcal{D}_{\mathcal{HF}}$ is a hyperdefuzzification operator satisfying $\mathcal{D}_{\mathcal{HF}}(\{x\}) = x$ for every $x \in [0,1]$. The alternative(s) with the smallest Q_j are considered the compromise solution(s).

Definition 2.2 (SuperHyperfuzzy VIKOR Method)**.** *Let $A = \{A_1, A_2, \ldots, A_J\}$ be a set of alternatives evaluated on n criteria. Suppose the performance of alternative A_j on criterion i is given by a* superhyperfuzzy number
$$\hat{f}_{ij} \in \mathcal{SHF}([0,1]),$$
where $\mathcal{SHF}([0,1])$ consists of mappings
$$\hat{f}_{ij} : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0,1]),$$
with $\tilde{\mathcal{P}}_n^*(X)$ being the family of all nonempty elements of the n-th powerset of a base set X. The SuperHyperfuzzy VIKOR method is defined analogously to the Hyperfuzzy VIKOR method (Definition 2.1) by replacing all hyperfuzzy operations ($\ominus$, $\odot$, $\oplus$, etc.) with their counterparts in the superhyperfuzzy algebra and by applying a superhyperfuzzy defuzzification operator $\mathcal{D}_{\mathcal{SHF}}$. That is, all steps (i) through (v) are carried out over the superhyperfuzzy structure.

Theorem 2.3 (Hyperfuzzy VIKOR Generalizes Fuzzy VIKOR)**.** *Assume that for all criteria i and alternatives j, the hyperfuzzy performance values are* degenerate, *that is,*
$$\hat{f}_{ij} = \{f_{ij}\}, \quad \text{with } f_{ij} \in [0,1].$$

Then, all hyperfuzzy operations reduce to standard arithmetic; in particular,

$$\{a\} \ominus \{b\} = \{a-b\}, \quad w_i \odot \{d\} = \{w_i\, d\}, \quad \bigoplus_i \{a_i\} = \left\{\sum_i a_i\right\},$$

and the hyperdefuzzification operator satisfies $\mathcal{D}_{\mathcal{HF}}(\{x\}) = x$. Consequently, the Hyperfuzzy VIKOR method reduces exactly to the fuzzy VIKOR method.

Proof. Since each $\hat{f}_{ij}$ is degenerate, we have

$$\hat{f}_{ij} = \{f_{ij}\} \quad \text{for all } i, j.$$

Then, the ideal and nadir values become

$$\hat{f}_i^* = \begin{cases} \{\max\{f_{ij}\}\}, & i \in I_b, \\ \{\min\{f_{ij}\}\}, & i \in I_c, \end{cases} \qquad \hat{f}_i^- = \begin{cases} \{\min\{f_{ij}\}\}, & i \in I_b, \\ \{\max\{f_{ij}\}\}, & i \in I_c. \end{cases}$$

Thus, the normalized difference for each criterion becomes

$$\hat{d}_{ij} = \begin{cases} \left\{ \dfrac{f_i^* - f_{ij}}{f_i^* - f_i^-} \right\}, & i \in I_b, \\ \left\{ \dfrac{f_{ij} - f_i^*}{f_i^- - f_i^*} \right\}, & i \in I_c, \end{cases}$$

which is exactly the normalized difference used in fuzzy VIKOR. Similarly, since

$$w_i \odot \hat{d}_{ij} = w_i \odot \{d_{ij}\} = \{w_i\, d_{ij}\} \quad \text{and} \quad \bigoplus_{i=1}^{n} \{w_i\, d_{ij}\} = \left\{ \sum_{i=1}^{n} w_i\, d_{ij} \right\},$$

we obtain

$$\hat{S}_j = \left\{ \sum_{i=1}^{n} w_i\, d_{ij} \right\} \quad \text{and} \quad \hat{R}_j = \left\{ \max_{1 \le i \le n} \{w_i\, d_{ij}\} \right\}.$$

The compromise measure

$$\hat{Q}_j = v\, \frac{\hat{S}_j \ominus \hat{S}^*}{\hat{S}^- \ominus \hat{S}^*} \oplus (1-v)\, \frac{\hat{R}_j \ominus \hat{R}^*}{\hat{R}^- \ominus \hat{R}^*}$$

reduces to

$$\hat{Q}_j = \left\{ v\, \frac{S_j - S^*}{S^- - S^*} + (1-v)\, \frac{R_j - R^*}{R^- - R^*} \right\}.$$

Applying the hyperdefuzzification operator,

$$Q_j = \mathcal{D}_{\mathcal{HF}}(\hat{Q}_j) = v\, \frac{S_j - S^*}{S^- - S^*} + (1-v)\, \frac{R_j - R^*}{R^- - R^*},$$

which is precisely the crisp compromise measure of the fuzzy VIKOR method. Hence, the Hyperfuzzy VIKOR method generalizes fuzzy VIKOR. □

Theorem 2.4 (SuperHyperfuzzy VIKOR Generalizes Hyperfuzzy VIKOR)**.** *Assume that for all criteria i and alternatives j, the superhyperfuzzy performance values are* degenerate *in the sense that*

$$\hat{f}_{ij} = \{f_{ij}\}, \quad f_{ij} \in [0,1],$$

when considered as elements of the superhyperfuzzy structure. Then, every superhyperfuzzy arithmetic operation reduces to the corresponding hyperfuzzy (and hence, standard) operation. In particular, the superhyperfuzzy VIKOR method reduces exactly to the Hyperfuzzy VIKOR method.

Proof. By Definition 2.2, each performance value $\hat{f}_{ij}$ in the SuperHyperfuzzy VIKOR method is a mapping

$$\hat{f}_{ij} : \tilde{\mathcal{P}}^*_n(X) \to \tilde{\mathcal{P}}([0,1]).$$

If these mappings are degenerate, then for every i, j and every element in the domain, the output is the singleton $\{f_{ij}\}$; that is,

$$\hat{f}_{ij} = \{f_{ij}\}.$$

Thus, all superhyperfuzzy operations (denoted by the same symbols as in the hyperfuzzy case) collapse to the corresponding hyperfuzzy operations. By Theorem 2.3, the hyperfuzzy VIKOR method further reduces to the standard fuzzy VIKOR method when degenerate. Therefore, the SuperHyperfuzzy VIKOR method, when applied to degenerate superhyperfuzzy data, reduces exactly to the Hyperfuzzy VIKOR method. □

Example 2.5. *Consider a decision problem with three alternatives*

$$A_1, \quad A_2, \quad A_3,$$

evaluated on two criteria. The performance of alternative A_j on criterion i is provided as a degenerate hyperfuzzy number:

$$\hat{f}_{ij} = \{f_{ij}\}, \quad \text{with } f_{ij} \in [0,1].$$

Suppose the following evaluation data are obtained:

Alternative	Criterion 1 (Benefit)	Criterion 2 (Cost)
A_1	$\{0.3\}$	$\{0.6\}$
A_2	$\{0.4\}$	$\{0.5\}$
A_3	$\{0.2\}$	$\{0.7\}$

We assume that:

- *Criterion 1 is a benefit criterion. Hence, higher values are preferred and the ideal for criterion 1 is the maximum value.*
- *Criterion 2 is a cost criterion. Hence, lower values are preferred and the ideal for criterion 2 is the minimum value.*

Let the set of benefit criteria be $I_b = \{1\}$ and the set of cost criteria be $I_c = \{2\}$. Assume the weights are equal:

$$w_1 = 0.5, \quad w_2 = 0.5,$$

and set the strategy parameter $v = 0.5$.

Step 1. Ideal and Nadir Values. *Since the hyperfuzzy numbers are degenerate, we simply use the crisp values:*

$$\text{For Criterion 1 (Benefit):} \quad f_{11} = 0.3, \quad f_{12} = 0.4, \quad f_{13} = 0.2.$$

Thus, the ideal (best) and nadir (worst) values for Criterion 1 are:

$$f_1^* = \max\{0.3,\, 0.4,\, 0.2\} = 0.4, \qquad f_1^- = \min\{0.3,\, 0.4,\, 0.2\} = 0.2.$$

For Criterion 2 (Cost), we have:

$$f_{21} = 0.6, \quad f_{22} = 0.5, \quad f_{23} = 0.7.$$

Since lower cost is better, the ideal and nadir are defined reversely:

$$f_2^* = \min\{0.6,\ 0.5,\ 0.7\} = 0.5, \qquad f_2^- = \max\{0.6,\ 0.5,\ 0.7\} = 0.7.$$

Step 2. Compute Normalized Differences. *For each alternative A_j and each criterion i we compute the normalized difference d_{ij}.*

Criterion 1 (Benefit): *For benefit criteria the normalized difference is defined as*

$$d_{1j} = \frac{f_1^* - f_{1j}}{f_1^* - f_1^-}.$$

Thus, we have:

$$\begin{aligned} d_{11} &= \tfrac{0.4-0.3}{0.4-0.2} = \tfrac{0.1}{0.2} = 0.5, \\ d_{12} &= \tfrac{0.4-0.4}{0.4-0.2} = \tfrac{0}{0.2} = 0, \\ d_{13} &= \tfrac{0.4-0.2}{0.4-0.2} = \tfrac{0.2}{0.2} = 1. \end{aligned}$$

Criterion 2 (Cost): *For cost criteria the normalized difference is defined as*

$$d_{2j} = \frac{f_{2j} - f_2^*}{f_2^- - f_2^*}.$$

Thus,

$$\begin{aligned} d_{21} &= \tfrac{0.6-0.5}{0.7-0.5} = \tfrac{0.1}{0.2} = 0.5, \\ d_{22} &= \tfrac{0.5-0.5}{0.7-0.5} = \tfrac{0}{0.2} = 0, \\ d_{23} &= \tfrac{0.7-0.5}{0.7-0.5} = \tfrac{0.2}{0.2} = 1. \end{aligned}$$

Step 3. Compute Utility (S_j) and Regret (R_j). *The group utility S_j of alternative A_j is the weighted sum of the normalized differences:*

$$S_j = w_1\, d_{1j} + w_2\, d_{2j}.$$

The individual regret R_j is the maximum weighted difference for alternative A_j:

$$R_j = \max\{w_1\, d_{1j},\ w_2\, d_{2j}\}.$$

Thus, for each alternative we obtain:

$$\begin{aligned} A_1: \quad & S_1 = 0.5 \times 0.5 + 0.5 \times 0.5 = 0.25 + 0.25 = 0.5, \\ & R_1 = \max\{0.5 \times 0.5,\ 0.5 \times 0.5\} = \max\{0.25,\ 0.25\} = 0.25; \\ A_2: \quad & S_2 = 0.5 \times 0 + 0.5 \times 0 = 0 + 0 = 0, \\ & R_2 = \max\{0.5 \times 0,\ 0.5 \times 0\} = 0; \\ A_3: \quad & S_3 = 0.5 \times 1 + 0.5 \times 1 = 0.5 + 0.5 = 1, \\ & R_3 = \max\{0.5 \times 1,\ 0.5 \times 1\} = \max\{0.5,\ 0.5\} = 0.5. \end{aligned}$$

Next, determine the best (ideal) and worst (nadir) values for S and R:

$$S^* = \min\{S_1,\ S_2,\ S_3\} = 0, \quad S^- = \max\{S_1,\ S_2,\ S_3\} = 1,$$

$$R^* = \min\{R_1,\ R_2,\ R_3\} = 0, \quad R^- = \max\{R_1,\ R_2,\ R_3\} = 0.5.$$

Step 4. Compute the Compromise Measure (Q_j). *The compromise measure for each alternative is given by*

$$Q_j = v\, \frac{S_j - S^*}{S^- - S^*} + (1 - v)\, \frac{R_j - R^*}{R^- - R^*}.$$

Substituting $v = 0.5$ and the values computed above:

$$\begin{aligned} Q_1 &= 0.5\,\tfrac{0.5-0}{1-0} + 0.5\,\tfrac{0.25-0}{0.5-0} \\ &= 0.5 \times 0.5 + 0.5 \times 0.5 \\ &= 0.25 + 0.25 = 0.5, \\ Q_2 &= 0.5\,\tfrac{0-0}{1-0} + 0.5\,\tfrac{0-0}{0.5-0} = 0, \\ Q_3 &= 0.5\,\tfrac{1-0}{1-0} + 0.5\,\tfrac{0.5-0}{0.5-0} \\ &= 0.5 \times 1 + 0.5 \times 1 \\ &= 0.5 + 0.5 = 1. \end{aligned}$$

Step 5. Ranking Alternatives. *Finally, we apply the hyperdefuzzification operator which, for degenerate hyperfuzzy numbers, simply extracts the unique element. Thus, the crisp values obtained are:*

$$Q_1 = 0.5, \quad Q_2 = 0, \quad Q_3 = 1.$$

Ranking the alternatives by increasing values of Q_j yields:

$$A_2 \prec A_1 \prec A_3.$$

That is, alternative A_2 is the best (with the lowest compromise measure), followed by A_1, and then A_3.

Note: *The same numerical procedure applies to the SuperHyperfuzzy VIKOR method. When the superhyperfuzzy numbers are degenerate (i.e. each is a singleton $\{f_{ij}\}$), all superhyperfuzzy operations reduce to their hyperfuzzy counterparts. Hence, the final ranking will be identical.*

2.2 Generalized DEMATEL Method

The definition of the Generalized DEMATEL Method, generalized from the perspective of Hyperfuzzy and Superhyperfuzzy approaches, is presented below.

Definition 2.6 (Hyperfuzzy DEMATEL Method). *Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of factors. In the* Hyperfuzzy DEMATEL *method the direct relationships among factors are assessed by experts using hyperfuzzy numbers. That is, the direct-relation matrix*

$$\hat{Z} = [\hat{z}_{ij}]_{n \times n}$$

has entries

$$\hat{z}_{ij} \in \mathcal{HF}([0,1]),$$

where $\mathcal{HF}([0,1])$ denotes the family of all nonempty subsets of $[0,1]$ (the hyperfuzzy numbers). The procedure is as follows:

1. ***Normalization.*** *Define a hyperfuzzy normalization constant*

$$s = \max_{1 \le i \le n} \Big\{ \bigoplus_{j=1}^{n} \hat{z}_{ij} \Big\},$$

where $\bigoplus$ is the hyperfuzzy addition. Then form the normalized direct-relation matrix

$$\hat{X} = [\hat{x}_{ij}], \quad \hat{x}_{ij} = \hat{z}_{ij} \oslash s,$$

with $\oslash$ denoting hyperfuzzy scalar division.

2. ***Total-Reaction Matrix.*** *Compute the total-relation matrix by*

$$\hat{T} = \hat{X} \odot (I \ominus \hat{X})^{-1},$$

where $\ominus$ denotes hyperfuzzy subtraction, $\odot$ denotes hyperfuzzy multiplication, and the inverse is defined in the hyperfuzzy-matrix sense.

3. ***Causal Indices.*** *For each factor s_i, define the hyperfuzzy sums:*

$$\hat{D}_i = \bigoplus_{j=1}^{n} \hat{t}_{ij}, \quad \hat{R}_i = \bigoplus_{j=1}^{n} \hat{t}_{ji}.$$

The causal influence of factor s_i is then given by the pair

$$(\hat{D}_i + \hat{R}_i,\ \hat{D}_i - \hat{R}_i).$$

A hyperdefuzzification operator $\mathcal{D}_{\mathcal{HF}}$ can be applied to yield crisp indices for further analysis.

Definition 2.7 (SuperHyperfuzzy DEMATEL Method)**.** *Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of factors. In the* SuperHyperfuzzy DEMATEL *method, the direct-relation evaluations are provided as* superhyperfuzzy numbers. *That is, the direct-relation matrix*

$$\hat{Z} = [\hat{z}_{ij}]_{n\times n}$$

has entries

$$\hat{z}_{ij} \in \mathcal{SHF}([0,1]),$$

where $\mathcal{SHF}([0,1])$ denotes the family of superhyperfuzzy numbers (i.e. mappings from the nonempty elements of the nth powerset of a base set into nonempty subsets of $[0,1]$). The SuperHyperfuzzy DEMATEL algorithm is defined analogously to Definition 2.6 by replacing all hyperfuzzy arithmetic operations with the corresponding superhyperfuzzy operations and applying a superhyperfuzzy defuzzification operator $\mathcal{D}_{\mathcal{SHF}}$.

Theorem 2.8 (Hyperfuzzy DEMATEL Generalizes Fuzzy DEMATEL)**.** *Assume that for every pair (i,j) the hyperfuzzy entry is* degenerate, *i.e.,*

$$\hat{z}_{ij} = \{z_{ij}\} \quad \text{with } z_{ij} \in [0,1].$$

Then the hyperfuzzy operations (addition, subtraction, scalar division, and matrix inversion) satisfy

$$\{a\} \oplus \{b\} = \{a+b\}, \quad \{a\} \ominus \{b\} = \{a-b\}, \quad \{a\} \oslash s = \{a/s\},$$

and the hyperdefuzzification operator obeys $\mathcal{D}_{\mathcal{HF}}(\{x\}) = x$. Consequently, the Hyperfuzzy DEMATEL method (Definition 2.6) reduces exactly to the standard fuzzy DEMATEL method.

Proof. If each hyperfuzzy number $\hat{z}_{ij}$ is degenerate in the sense that $\hat{z}_{ij} = \{z_{ij}\}$, then by the definition of hyperfuzzy arithmetic the following equalities hold:

$$\{a\} \oplus \{b\} = \{a+b\}, \quad \{a\} \ominus \{b\} = \{a-b\}, \quad \{a\} \oslash s = \{a/s\}.$$

Thus, the normalized matrix becomes

$$\hat{x}_{ij} = \hat{z}_{ij} \oslash s = \{z_{ij}/s\},$$

and every subsequent arithmetic operation in the procedure (including the computation of the total-relation matrix and the summations to obtain $\hat{D}_i$ and $\hat{R}_i$) produces degenerate hyperfuzzy numbers corresponding exactly to the values that would be obtained in the standard fuzzy DEMATEL method. Applying the hyperdefuzzification operator $\mathcal{D}_{\mathcal{HF}}$, we recover the crisp numbers. Hence, the Hyperfuzzy DEMATEL method is a proper generalization of the fuzzy DEMATEL method. □

Theorem 2.9 (SuperHyperfuzzy DEMATEL Generalizes Hyperfuzzy DEMATEL)**.** *Assume that for every pair* (i, j) *the superhyperfuzzy performance value is* degenerate, *i.e.*,

$$\hat{z}_{ij} = \{z_{ij}\} \quad \textit{with } z_{ij} \in [0, 1],$$

when considered as elements in the superhyperfuzzy structure. Then, all superhyperfuzzy operations reduce to the corresponding hyperfuzzy operations. In particular, the SuperHyperfuzzy DEMATEL method (Definition 2.7) reduces exactly to the Hyperfuzzy DEMATEL method.

Proof. By the definition of a superhyperfuzzy number, each entry

$$\hat{z}_{ij} : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0, 1])$$

assigns a nonempty subset of $[0, 1]$. If the superhyperfuzzy numbers are degenerate, then for every (i, j) we have

$$\hat{z}_{ij} = \{z_{ij}\}.$$

Thus, every arithmetic operation in the superhyperfuzzy framework (denoted using the same symbols $\oplus$, $\ominus$, $\oslash$, etc.) reduces to the corresponding hyperfuzzy arithmetic:

$$\{a\} \oplus \{b\} = \{a + b\}, \quad \{a\} \ominus \{b\} = \{a - b\}, \quad \{a\} \oslash s = \{a/s\}.$$

Therefore, following Definition 2.7, the entire procedure of SuperHyperfuzzy DEMATEL collapses to that of Hyperfuzzy DEMATEL. Applying the superhyperfuzzy defuzzification operator $\mathcal{D}_{\mathcal{SHF}}$, which in the degenerate case satisfies $\mathcal{D}_{\mathcal{SHF}}(\{x\}) = x$, we recover the crisp indices identical to those from Hyperfuzzy DEMATEL. Hence, SuperHyperfuzzy DEMATEL is a proper generalization of Hyperfuzzy DEMATEL. □

Example 2.10. *Assume that a group of experts evaluates the causal influence among three factors*

$$s_1, \quad s_2, \quad s_3.$$

In the classic fuzzy DEMATEL method, the direct influence is quantified by the matrix

$$Z = \begin{bmatrix} 0 & 0.5 & 0.4 \\ 0.6 & 0 & 0.7 \\ 0.3 & 0.4 & 0 \end{bmatrix}.$$

Step 1: Normalization.
For each factor s_i, compute the row sum:

$$\begin{aligned} \textit{Row } 1: &\quad 0 + 0.5 + 0.4 = 0.9, \\ \textit{Row } 2: &\quad 0.6 + 0 + 0.7 = 1.3, \\ \textit{Row } 3: &\quad 0.3 + 0.4 + 0 = 0.7. \end{aligned}$$

The normalization constant is

$$s = \max\{0.9,\ 1.3,\ 0.7\} = 1.3.$$

Thus, the normalized direct-relation matrix is

$$X = \frac{1}{1.3} Z = \begin{bmatrix} 0 & 0.3846 & 0.3077 \\ 0.4615 & 0 & 0.5385 \\ 0.2308 & 0.3077 & 0 \end{bmatrix}.$$

Step 2: Computation of the Total-Relation Matrix.

The total-relation matrix is defined by

$$T = X\,(I - X)^{-1},$$

where I is the 3×3 identity matrix. (For a convergent series, one may also write $T = X + X^2 + X^3 + \cdots$.) For simplicity, we approximate T by summing the first two terms:

$$T \approx X + X^2.$$

First, we compute X^2. Denote $X = [x_{ij}]$ where:

$$x_{11} = 0,\ x_{12} = 0.3846,\ x_{13} = 0.3077;$$

$$x_{21} = 0.4615,\ x_{22} = 0,\ x_{23} = 0.5385;$$

$$x_{31} = 0.2308,\ x_{32} = 0.3077,\ x_{33} = 0.$$

Then, for example, the $(1, 1)$ element of X^2 is:

$$(X^2)_{11} = x_{12}x_{21} + x_{13}x_{31} \approx (0.3846)(0.4615) + (0.3077)(0.2308) \approx 0.1777 + 0.0710 = 0.2487.$$

Similarly, one obtains

$$X^2 \approx \begin{bmatrix} 0.2487 & 0.0946 & 0.2077 \\ 0.1242 & 0.3434 & 0.1420 \\ 0.1420 & 0.0889 & 0.2367 \end{bmatrix}.$$

Thus, an approximate total-relation matrix is

$$T \approx X + X^2 = \begin{bmatrix} 0.2487 & 0.3846 + 0.0946 & 0.3077 + 0.2077 \\ 0.4615 + 0.1242 & 0 + 0.3434 & 0.5385 + 0.1420 \\ 0.2308 + 0.1420 & 0.3077 + 0.0889 & 0 + 0.2367 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.2487 & 0.4792 & 0.5154 \\ 0.5857 & 0.3434 & 0.6805 \\ 0.3728 & 0.3966 & 0.2367 \end{bmatrix}.$$

Step 3: Calculation of Causal Indices.

Define for each factor s_i:

$$D_i = \sum_{j=1}^{3} t_{ij}, \quad R_i = \sum_{j=1}^{3} t_{ji}.$$

Then,

$$D_1 = 0.2487 + 0.4792 + 0.5154 \approx 1.2433,$$
$$D_2 = 0.5857 + 0.3434 + 0.6805 \approx 1.6096,$$
$$D_3 = 0.3728 + 0.3966 + 0.2367 \approx 1.0061.$$

Similarly, the column sums are:

$$R_1 = 0.2487 + 0.5857 + 0.3728 \approx 1.2072,$$
$$R_2 = 0.4792 + 0.3434 + 0.3966 \approx 1.2192,$$
$$R_3 = 0.5154 + 0.6805 + 0.2367 \approx 1.4326.$$

The prominence *of each factor is* $D_i + R_i$*:*

$$\begin{aligned} s_1 &: \ 1.2433 + 1.2072 \approx 2.4505, \\ s_2 &: \ 1.6096 + 1.2192 \approx 2.8288, \\ s_3 &: \ 1.0061 + 1.4326 \approx 2.4387. \end{aligned}$$

The net causal effect *of each factor is* $D_i - R_i$*:*

$$\begin{aligned} s_1 &: \ 1.2433 - 1.2072 \approx 0.0361, \\ s_2 &: \ 1.6096 - 1.2192 \approx 0.3904, \\ s_3 &: \ 1.0061 - 1.4326 \approx -0.4265. \end{aligned}$$

These results indicate that s_2 *is the most influential (a driving or cause factor), while* s_3 *is primarily affected (an effect factor).*

Hyperfuzzy DEMATEL Procedure.

In the Hyperfuzzy DEMATEL method, experts provide their judgments as hyperfuzzy numbers. Suppose the hyperfuzzy direct-relation matrix is given by

$$\hat{Z} = \begin{bmatrix} \{0\} & \{0.5\} & \{0.4\} \\ \{0.6\} & \{0\} & \{0.7\} \\ \{0.3\} & \{0.4\} & \{0\} \end{bmatrix}.$$

Here, each entry is a degenerate hyperfuzzy number (i.e. a singleton set). Following Definition 2.6, all hyperfuzzy arithmetic operations (normalization, addition, subtraction, etc.) reduce to their conventional counterparts. Therefore:

- ***Normalization:*** *The hyperfuzzy normalization constant is*

$$s = 1.3,$$

  and the normalized hyperfuzzy matrix is

$$\hat{X} = \frac{1}{1.3}\,\hat{Z} = \begin{bmatrix} \{0\} & \{0.3846\} & \{0.3077\} \\ \{0.4615\} & \{0\} & \{0.5385\} \\ \{0.2308\} & \{0.3077\} & \{0\} \end{bmatrix}.$$

- ***Total-Reaction Matrix and Causal Indices:*** *Proceeding with the hyperfuzzy operations (which, because of degeneration, match standard arithmetic) yields a total-relation matrix*

$$\hat{T} \approx \begin{bmatrix} \{0.2487\} & \{0.4792\} & \{0.5154\} \\ \{0.5857\} & \{0.3434\} & \{0.6805\} \\ \{0.3728\} & \{0.3966\} & \{0.2367\} \end{bmatrix},$$

  and hyperfuzzy row and column sums:

$$\hat{D}_1 = \{1.2433\}, \quad \hat{D}_2 = \{1.6096\}, \quad \hat{D}_3 = \{1.0061\},$$

$$\hat{R}_1 = \{1.2072\}, \quad \hat{R}_2 = \{1.2192\}, \quad \hat{R}_3 = \{1.4326\}.$$

Applying a hyperdefuzzification operator $\mathcal{D}_{\mathcal{HF}}$ (which in the degenerate case simply extracts the singleton value) recovers the crisp results:

$$D_1 \approx 1.2433, \quad D_2 \approx 1.6096, \quad D_3 \approx 1.0061,$$

$$R_1 \approx 1.2072, \quad R_2 \approx 1.2192, \quad R_3 \approx 1.4326.$$

Thus, the causal diagram for the factors, expressed as the coordinates

$$(s_i^{\textit{prominence}},\ s_i^{\textit{net effect}}) = (D_i + R_i,\ D_i - R_i),$$

is

$$s_1 : (2.4505,\ 0.0361), \quad s_2 : (2.8288,\ 0.3904), \quad s_3 : (2.4387,\ -0.4265),$$

which is identical to the classic fuzzy DEMATEL result.

Similarly, if one were to construct the SuperHyperfuzzy DEMATEL matrix with degenerate superhyperfuzzy numbers, the same detailed steps would hold, and the final crisp indices would also be identical.

3. Conclusion

In this paper, we introduced novel formulations of Hyperfuzzy VIKOR and Hyperfuzzy DEMATEL that extend the classical Fuzzy VIKOR and Fuzzy DEMATEL methods. We further developed corresponding extensions based on the SuperHyperfuzzy Set paradigm, laying the groundwork for richer, hierarchically structured decision-making models.

As future work, we aim to explore the development of decision-making methods utilizing HyperGraphs[48–50] and SuperHyperGraphs[51–54], implement corresponding computational tools, and investigate further extensions based on HyperNeutrosophic Sets[7, 55, 56] and HyperPlithogenic Sets [57, 58].

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Conflict of Interest

The author declares that there are no conflicts of interest related to this manuscript.

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