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A Robust Circular Complex Intuitionistic Fuzzy Framework for Optimizing Agricultural Robot Decision-Making Under Uncertain Environmental Conditions

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ABSTRACT

Agricultural robotic systems frequently operate in highly dynamic and uncertain environmental conditions, where incomplete sensor readings, weather variability, terrain irregularities, and crop-state ambiguity significantly affect decision-making performance. To address these challenges, this study introduces the concept of Circular Complex Intuitionistic Fuzzy Sets (CrC-IFS) as an advanced mathematical tool for modeling uncertainty in agricultural robot decision-making processes. The Circular Complex Intuitionistic Fuzzy Set extends classical complex intuitionistic and circular intuitionistic fuzzy structures by incorporating enhanced higher-order membership flexibility for representing imprecise and multidimensional environmental information.

To strengthen uncertainty modeling capability, refined algebraic operational laws for CrC-IFS are developed, including direct sum, direct product, and scalar multiplication based on generalized t-norms and t-conorms. Furthermore, circular complex intuitionistic fuzzy weighted and ordered weighted aggregation operators are proposed to integrate multiple environmental and operational criteria, such as terrain conditions, obstacle density, energy consumption, crop maturity levels, and weather fluctuations.

Building upon these theoretical developments, a robust multi-criteria decision-making framework is constructed to optimize agricultural robot strategies, enabling systematic prioritization of navigation paths, task allocation, harvesting schedules, and adaptive control actions. The results demonstrate that the proposed framework enhances decision robustness, improves operational efficiency, and supports intelligent autonomous behavior under highly uncertain agricultural field conditions.

Keywords:

Circular Complex Intuitionistic Fuzzy set; Complex Intuitionistic Fuzzy Set; Multi-Criteria Decision Making.

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1. Introduction

Multi-attribute decision-making (MADM) is a fundamental component of intelligent agricultural robotic systems, where autonomous agents must select the most appropriate operational strategy, navigation path, or task execution plan from a finite set of alternatives based on multiple environmental, technical, and performance-related criteria. In smart farming environments, decisions are often influenced by inputs obtained from various sensors, expert evaluations, and real-time field data. Therefore, effective MADM frameworks enable agricultural robots to systematically evaluate competing alternatives under uncertain environmental conditions, ensuring optimized performance, resource efficiency, and adaptive field operations.

1.1 *Short review of T-spherical fuzzy set*

The concept of fuzzy set (FS) was initiated by Zadeh in 1965 **[1]** to help manage the modeling of real-world issues involving ambiguous data. FS theory is particularly a helpful tool for modeling ambiguity, it may be used to model and solve problems in a wide range of fields, including data mining, grouping, and medical research. An FS is defined by a membership function (MF) η from a collection of the universe's objects or elements to the interval [0,1], where the MF η, belong to the range [0,1]. Subsequently, Atanassov **[2]** extended FS by adding a non-membership function (NMF) ψ to present the idea of intuitionistic FS (IFS) ensuring the addition of MF and NMF satisfies $0 \leq \eta + \psi \leq 1$. However, in cases involving the pairs like $(0.7, 0.6)$, the concept of IFS did not satisfy that is, $0.7 + 0.6 = 1.3 > 1$. Therefore, Yager **[3]** introduced the Pythagorean FS $(PyFS)$, applying the new condition where $0 \leq \eta^2 + \psi^2 \leq 1$. However, the case $(0.8, 0.9)$ did not satisfy the condition of $PyFS$, that is, $0.8^2 + 0.9^2 > 1$. The Senapat [5] developed the concept Intuitionistic fuzzy sets, applying the new condition where $0 \leq \eta^3 + \psi^3 \leq 1$. However, the same case $(0.8, 0.9)$ also did not satisfy the condition of $PyFS$, that is, $0.8^3 + 0.9^3 > 1$. In order to remove this kind of restriction, Yager **[4]** developed the concept called q-Rung Orthopair $FS(q-ROFS)$, with the restriction $0 \leq \eta^q + \psi^q \leq 1$ where $q \geq 1$, which cover the failure situations.

Real-world decision-making (DM) can produce additional responses for a choice index, like neutral, abstention, negative, and positive. For this, Cuong **[6]** initiated the concept of Picture FS $(PiFS)$, which handle the membership value (MV) η $(0 \leq \eta \leq 1)$, non membership value (NMV) η $(0 \leq \eta \leq 1)$, and neutral value (NV) ϕ $(0 \leq \phi \leq 1)$, with the condition $0 \leq \eta + \psi + \phi \leq 1$. However, sometimes, it is challenging to accept certain restrictions, that is, $0.2 + 0.8 + 0.1 = 1.1 > 1$ does not belong to [0,1]. The Spherical FS (SFS) initiated by Mahmood et al. **[7]**, which is more effective as compared to $PiFS$ and IFS, with the condition that the sum of squares the MV, NMV and NV is less than or equal to 1, that is, $0 \leq \eta^2 + \psi^2 + \phi^2 \leq 1$. Sometimes sum of their MVs is greater than 1. However, in case if the MV, NMV and NV are $(0.7, 0.6, 0.9)$ then only the square not enough because the condition $0 \leq \eta^2 + \psi^2 + \phi^2 \leq 1$ is not satisfied, that is, $7^2 + 0.8^2 + 0.1^2 = 1.14 > 1$. To handle such kind of situations, Ullah et al. **[8]** initiated the T-spherical FS $(TSFS)$ as the extension of SFS, with the condition that $0 \leq \eta^q + \psi^q + \phi^q \leq 1$ where $q > 1$.. All of these developments provide useful tools for simulation, decision-making, environmental sustainabil ity and the advancement of the theoretical underpinnings and real-world implementations of intelligent systems in educational contexts. Fuzzy techniques to additive manufacturing, pre qualiÖcation evaluation, and sustainability are among the critical applications that this study tackles.

1.2 Short review of complex T-spherical fuzzy set

Forming decisions in real-life scenarios, decision-makers (DMs) encounter additional barriers to choosing the most viable option among a range of achievable opportunities as the systems become increasingly complex. Although it is one of the major challenges, it does not mean that it is impossible to reach one goal. Most corporations have been experiencing the dilemmas of establishing objectives, shaping perspectives and motivating workforces. The decisions by an organization committee or individuals, are therefore based on multiple goals being achieved at the same time. It is a requirement of every DM to come up with the most appropriate response to every realistic implications in real world issues through use of criteria that allow optional responses. As a result, DMs are becoming more committed to creating more dependable and practical techniques for identifying the best solutions. As per the research mentioned before, comparable methods have limitations and cannot capture the degree of uncertainty in the information and its variations throughout a specific time frame. Therefore, the reseachers arise the question that what would happen if we changed the FSs range into a unit disk in a complex plane. Therefore, Ramot et al. [9] initiated the Complex FS (CFS). The fundamental concept of CFS is to extend the membership range for $[0,1]$ to the complex plane. The MF $\eta = re^{i\theta}$, where r is called amplitue term and it value belong to the interval [0,1], θ is say to be phase term or periodic term and it is belong to $[0,2\pi]$. CFS only deal the MF, however, in certain cases, utilizing only the complex-valued MF $(CVMF)$ to solve the problem is either impossible or extremely difficult. To overcome such kind of situations, Alkouri and Salleh [10] initiated the Complex IFS $(CIFS)$, adding complex-valued non-membership function $(CVNMF)$, the idea of IFS is expanded in the comprehensive mathematical framework known as a $CIFS$, which are defined by the $CVMF$ $\eta_C = r_c.e^{iw_{\eta_c}}$ and $CVNMF$ $\psi_c = k_c.e^{iw_{\psi_c}}$, where $r_c,\ k_c,\ w_{\eta_c}$ and w_{ψ_c} belong to the interval [0,1]. The restriction of the $CIFS$ is that the sum of $CVMF$ and $CVNMF$ is equal to or less than 1, that is, $|\eta_C + \psi_c| \leq 1$. It is well $CIFS$ have two membership functions $(CVMF$ and $CVNMF)$, therefore, it could be challenging to explain some complicated $CIFS$ in fuzzy data. If the decision maker give $0.5.e^{2\pi i(0.42)}$ for the MV and $0.7.e^{2\pi i(0.79)}$ for the NMV, the $CIFS$ condition is not satisfied, because is, $0.5 + 0.7 = 1.2 > 1$ and $0.42 + 0.79 = 1.21 > 1$. Therefore, Ullah et al. [11] developed the complex pythagorean FS $(CPyFS)$. The $CPyFS$ has the property the sum of squares $CVMF$ and $CVNMF$ is less than or equal to 1. The $CPyFS$ is more effective as compared to $CIFS$, that is, $0.5^2 + 0.7^2 = 0.74 < 1$ and $0.42^2 + 0.79^2 = 1.21 > 1$, in $CIFS$ and $CPyFS$, the same scenario arises when the decision-maker provides such information, which don't fulfill the requirements of $CIFS$ and $CPyFS$. For example, the decision maker take the complex valued membership degree $0.8.e^{2\pi i(0.92)}$ and complex valued non-membership degree $0.7.e^{2\pi i(0.81)}$, the $CIFS$ and $CPyFS$ do not satisfy the conditions i.e. $0.8+0.7 = 1.5 > 1,\ 0.8^2+0.7^2 = 1.13 > 1$ and $0.92 + 0.81 = 1.73,\ 0.92^2 + 0.81^2 = 1.502 > 1$ and complex Intuitionistic fuzzy set [12] also not satisfy. However, Liu et al. [13] introduced the Complex $q - ROFS$ $(Cq - ROFS)$. Their most notable condition is that the sum of qth power of $CVMF$ and qth power of $CVNMF$ is equal to or less than 1, that is, $0.8^3 + 0.7^3 = 0.855$. An essential method for handling ambiguous and challenging data is the suggested $Cq - ROFS$, which may then be used to address multi-attribute decision making $(MADM)$ issues.

$Cq - ROFS$ is unable to deal data that is neglected such as $Cq - ROFS$ only study of the $CVMF$ and $CVNMF$ but it cannot study of complex neutral function (CNF). For this, Akram et al. [14] introduced the Complex $PiFS$, the amplitude terms $(\eta, \psi, \phi) \in [0,1])$ and phase terms $(\theta, \delta, \gamma \in [0,2]),\ CPiFS$ satisfied the conditions $0 \leq \eta, \psi, \phi \leq 1$ and $0 \leq \theta+\delta+\gamma \leq 2$, respectively. However, in cases involving the pairs of amplitude term like (0.5,0.6,0.2) the condition of $CPiFS$ is not satisfy. Additionally, the complex SFS $(CSFS)$ was defined by Akram et al. [15] as a generalization of $CPiFS$. However, Ali et al. [16] introduced the complex $T - SFS$ $(CT - SFS)$,which cover the failure situations.

1.3 Short review of cricular q-ROF set

Another question raised by many researchers is what would happen if we transforme the range of Intuitionistic Fuzzy Sets $(IFSs)$ into a circular region? Therefore Atanassov **[17]** initiated the concept of Circular Intuitionistic Fuzzy Sets $(CIFSs)$. Furthermore, Bozyigit et al. **[18]** developed the circular PFS $(Cr - PFS)$ as an extension of $Cr - IFS$, in which $Cr - PFS$ contains the membership degree, represented by η, and the non-membership degree is shown by ψ under the condition of $0 \leq \eta^2 + \psi^2 \leq 1$, with the radius $R \in [0, 1]$ of a circle around the point (η, ψ) on the plane. Several studies have explored fuzzy MCDM approaches to evaluate technological adoption within supply chains. Recently, Yusoff et al. **[19]** extended $Cr - PFS$ to circular $q - ROFSs$ $(Cirq - ROFSs)$, which have a condition of $0 \leq \eta^q + \psi^q \leq 1$, $q \geq 1$ with a radius of a circle around the point (η, ψ) with $R \in [0, 1]$, and then gave some basic algebraic proper. However, Zeeshan et al. **[21]** define identification of the novel technique of Cirq-ROFSs with their flexible properties, such as algebraic laws and Dombi laws.
Recent advancements in multi-criteria decision-making (MCDM) under uncertainty have significantly enhanced business and engineering decision processes. For instance, Ullah et al. [30] introduced a business-oriented stock market decision framework using circular complex picture fuzzy sets integrated with the CRITIC–WASPAS method. Similarly, Liu et al. [31] developed a prospect-regret based three-way decision model utilizing q-rung orthopair fuzzy preference relations to address energy crisis problems. In the domain of sustainable development, Liu et al. [32] analyzed green building evaluation using CODAS and WASPAS methods under circular linguistic T-spherical fuzzy Hamy mean aggregation operators. Moreover, Ali et al. [33] explored artificial intelligence applications in medical practice through CRITIC-TOPSIS based on λ (pq)-cubic quasi rung orthopair fuzzy robust aggregation operators.

1.4 The main motivations

Overall, circular complex Intuitionistic fuzzy set (CrC-IFFS) is the extensive generalization of Intuitionistic fuzzy set. Nonetheless, some situations cannot be managed well by C-IFS. The proposed within the framework of this paper is the new method of Cricular Complex Intuitionistic fuzzy set Fuzzy Set (CrC-IFS) whose operational laws include CrC-IFS a combination of the degrees of membership, abstinence, and non-membership with a condition in which the total of power of the real part (as well as imaginary part) of the membership, abstinence, and non-membership grades is not more than a unit interval.

MCDM methods are progressively gaining popularity as future devices of evaluating and solving complex real-time dilemmas owing to their inherent ability to evaluate a large number of choices by a number of variables with the aim of possibly selecting the most suitable choice. The challenges in MCDM have several distinctiveness like the existence of more than one non-commensurable and conflicting criteria, the criteria have different units of measurement, and the existence of relatively dissimilar options, as well. These decision-making issues characterize multidimensional situations and are being addressed using the MCDM techniques. The MCDM techniques are aimed mostly at the analysis and prioritization of the available alternatives. This is so many times because MCDM techniques can give different results (i.e. the same alternatives are ranked differently depending on which techniques are applied). This is due to the numerous mathematical artifacts in which the methodologies under discussion make use of them. The WASPAS technique is a peculiar blend of two popular MCDM techniques i.e. weighted sum model (WSM) and weighted product model (WPM). Its use involves first developing decision/evaluation matrix, $X = [x_{ij}]_{mxn}$ where the x_{ij} is the performance of i_{th} alternative against j_{th} criterion where m and n are the number of alternatives and number of criteria respectively. The multiple options that are now feasible are ranked in terms of the value of Q and the top alternative is that with the maximum value of Q. Table 1 presents the exact limitations of the previous studies against the suggested strategy.

Table 1
Comparison of the proposed model with extant models in the literature

Concept	**MD**	**NMD**	**CMD**	**CNMD**	**Radius**	$\boldsymbol{\lambda}$	$\boldsymbol{\beta}$
FS	Yes	No	No	No	No	No	No
IFS	Yes	Yes	No	No	No	No	No
PyFS	Yes	Yes	No	No	No	No	No
q-ROFS	Yes	Yes	No	No	No	No	No
PiFS	Yes	Yes	No	No	No	No	No
SFS	Yes	Yes	No	No	No	No	No
T-SFS	Yes	Yes	No	No	No	No	No
CrIFS	Yes	Yes	No	No	Yes	No	No
CrPyFS	Yes	Yes	No	No	Yes	No	No
Crq-ROFS	Yes	Yes	No	No	Yes	No	No
CFS	Yes	No	Yes	No	No	No	No
CIFS	Yes	Yes	Yes	Yes	No	No	No
CPyFS	Yes	Yes	Yes	Yes	No	No	No
Cq-ROFS	Yes	Yes	Yes	Yes	No	No	No
CPiFS	Yes	Yes	Yes	Yes	No	No	No
CSFS	Yes	Yes	Yes	Yes	No	No	No
CT-SFS	Yes	Yes	Yes	Yes	No	No	No
Proposed CrC-IFS	Yes	Yes	Yes	Yes	Yes	Yes	Yes

We found from the aforementioned investigation that the following are the main issues that all experts have:

(i) On the basis of CrC-IFSs, how should novel operational laws be drafted?
(ii) On the basis of novel operational laws, how may new operators be developed?
(iii) How can all actions be ranked according to the developed operators?

1.5 *Novelty and main contributions*

This research is intended to establish a rational and intellectual method of supporting a decision so that the most appropriate alternative can be adopted out of a number of alternative sources. The integration of algebraic complex operational rules into the CrC-IFS environment will allow one to use the CrC-IFS arithmetic and geometric mean aggregation operators which will ensure the efficiency of the conceptual framework.

The key achievements and objectives of this article are as follows:

1. It is more competent, comprehensive and reliable than the present day conception such as CrC-IFS in the aspect of uncertain data to be dealt with in the decision making process. Also, no prior

studies have examined the relationship between CrC-IFS situations and the use of arithmetic and geometric mean aggregation operators based on algebraic complex operations laws. Thus, it is important to improve geometrical and arithmetic mean of aggregation operators based on complex operational principles to solve MCDM problems in CrC-IFS cases.

2. An interesting concept to managing data in three dimensions in one set is the arithmetic and geometric mean aggregation operators pursuant to setting CrC-IFS that uses complex operational rules. Thus, this research aims at providing Crcular Complex-IFS weighted arithmetic mean aggregation operator (CrC-IFWAM), Crcular Complex FFS ordered weighted arithmetic mean aggregation operator (CrC-IFOWAM), Crcular Complex IFS weighted geometric mean aggregation operator (CrC-IFWGM) and Crcular Complex FFS ordered weighted geometric mean aggregation operator (CrC-IFOWGM).

3. The relations defining between these operators are emphasized to deal with some of their features such as being bounded, idempent and monotonic.

4. To develop two different, innovative approaches founded on the CrC-IFWAM and CrC-IFWGM.

5. An illustrative representation of the given approach is provided to make the proposed procedure even more understandable and clear. The recommended modelling technique is graphically presented by applying a flowchart in visualising the desired process.

6. To give examples of application that can prove the feasibility and reliability of the proposed techniques. Also, the comparison of the proposed techniques with the existing methods will show that the proposed techniques are superior and that the aggregation process will be more flexible as the arithmetic and geometric mean aggregation operators are used in accordance with CrC-IFS conditions and the complicated rules of operation.

1.6 The structure of this paper

As shown below, this article is structured as follows: Section 2 deals with the construct of complex operational laws based CrC-IFWAM aggregation operator, CrC-IFOWAM aggregation operator (CrC-IFOWAM). Section 3 discusses the concept of CrC-IFWGM aggregation operator and CrC-IFOWGM aggregation operator in the framework of Circular Complex Intuitionistic fuzzy sets and their properties. Section 4 proposes an innovative method of decision-making through the latest methods that rely on the CrC-IFWAM and CrC-IFWGM. Besides, Section 5 includes an example to illustrate the worth of the proposed strategy to choose. Section 6 defines the comparability and sensitivity analysis which show the logic and stability of the proposed technique. Section 7 gives a conclusion to the article.

2. Circular Complex Intuitionistic fuzzy weighted arithmetic mean aggregation operators

We provide definitions of Circular Complex Intuitionistic fuzzy operational laws for CrC-IFNs in the following section. Following them, several aggregation (circular complex Intuitionistic fuzzy weighted arithmetic mean aggregation operator (CrC-IFWAM), circular complex PiSF ordered weighted arithmetic mean aggregation operator (CrC-IFOWAM)) operators based on circular complex Intuitionistic fuzzy operational laws will be created.

Definition 1. *[9] A CFS A is defined as:*

$$A = \{(x, \eta(x)) \mid x \in X\}$$

where $\eta(x) = \eta(x)e^{i2\pi\omega(x)}$ *denotes the grade of complex-valued truth with a condition:* $0 \leq \eta(x), \omega(x) \leq 1$.

2.1 Proposed Circular Complex Intuitionistic fuzzy Sets

One aim of this study is to explore the novel approach of CrC-IFSs and their operational laws. These operational laws are also verified with the help of a numerical example.

Definition 2. *A CrC-IFSP is defined as:*

$$P = \{(x, \eta(x), \phi(x), r(x)) \mid x \in X\}$$

where $\eta(x) = \eta_{C_1}e^{i2\pi\eta^{+im}_{C_1}}$, $\phi(x) = \phi_{C_1}e^{i2\pi\phi^{+im}_{C_1}}$, *and* $r(x) = r_{C_1}e^{i2\pi r^{+im}_{C_1}}$ *denote the membership degree, non-membership and radius with the conditions:* $0 \leq \eta_{C_1} + \phi_{C_1} \leq 1$ *and* $0 \leq (\eta^{i}_{C_1} + \phi^{i}_{C_1}) \leq 1$. *Additionally, the term* $H(x) = Re^{i2\pi\omega_R(x)}$ *such that* $R = (1 - (\eta_{C_1} + \phi_{C_1}))$ *and* $\omega_R(x) = \left(1 - (\eta^{i}_{C_1} + \phi^{i}_{C_1})\right)$ *expresses the complex hesitancy grade of* x. *Moreover,*
$P = \left(\eta_{C_1}e^{i2\pi\eta^{+im}_{C_1}}, \phi_{C_1}e^{i2\pi\phi^{+im}_{C_1}}, r_{C_1}e^{i2\pi r^{+im}_{C_1}}\right)$ *is called a CrC-IFN.*

Definition 3. *For any CrC-IFN.* $P_1 = \left(\eta_{C_1}e^{i2\pi\eta^{+im}_{C_1}}, \phi_{C_1}e^{i2\pi\phi^{+im}_{C_1}}, r_{C_1}e^{i2\pi r^{+im}_{C_1}}\right)$, *the score and accuracy functions are defined by*

$$SC\left(P_1\right) = \frac{1}{8}\left\{\ (\eta_{C_1}) + \left(\eta^{im}_{C_1}\right) - (\phi_{C_1}) - \left(\phi^{im}_{C_1}\right) + (r_{C_1}) + \left(r^{im}_{C_1}\right)\ \right\}$$

and

$$AC\left(P_1\right) = \frac{1}{8}\left\{\ (\eta_{C_1}) + \left(\eta^{im}_{C_1}\right) + (\phi_{C_1}) + \left(\phi^{im}_{C_1}\right) + (r_{C_1}) + \left(r^{im}_{C_1}\right)\ \right\}$$

where $SC\left(P_1\right) \in [-1, 1]$ *and* $AC\left(P_1\right) \in [0, 1]$.

Definition 4. *Let* $P_1 = \left(\eta_{C_1}e^{i2\pi\eta^{+im}_{C_1}}, \phi_{C_1}e^{i2\pi\phi^{+im}_{C_1}}, r_{C_1}e^{i2\pi r^{+im}_{C_1},}\right)$,
$P_2 = \left(\eta_{C_2}e^{i2\pi\eta^{+im}_{C_1}}, \phi_{C_2}e^{i2\pi\phi^{+im}_{C_2}}, r_{C_1}e^{i2\pi r^{+im}_{C_2},}\right)$ *be two Cq -ROFNs. Then*
(1) *if* $SC\left(P_1\right) > SC\left(P_2\right)$, *then* $P_1 > P_2$,
(2) *if* $SC\left(P_1\right) = SC\left(P_2\right)$ *then*
(i) *if* $AC\left(P_1\right) > AC\left(P_2\right)$, *then* $P_1 > P_2$,
(ii) *if* $AC\left(P_1\right) = AC\left(P_2\right)$, *then* $P_1 = P_2$.

2.2 Algebraic Circular Complex Intuitionistic fuzzy operational laws

The new direct sum, direct product, and scalar multiplication operations are defined for CrC-IFSs in this subsection.

Definition 5. *Let* $P_1 = \left(\eta_{C_1}e^{i2\pi\eta^{+im}_{C_1}}, \phi_{C_1}e^{i2\pi\phi^{+im}_{C_1}}, r_{C_1}e^{i2\pi r^{+im}_{C_1},}\right)$,
$P_2 = \left(\eta_{C_2}e^{i2\pi\eta^{+im}_{C_1}}, \phi_{C_2}e^{i2\pi\phi^{+im}_{C_2}}, r_{C_1}e^{i2\pi r^{+im}_{C_2},}\right)$ *be two CrC -IFNs. Then we define the following algebraic Circular Complex Intuitionistic fuzzy operational laws:*

$$(\mathrm{i})\ P_1 \oplus^1 P_2 = \left(\begin{array}{c} \left(\frac{1}{2}\left((\eta_{C_1}) + (\eta_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(\eta^{im}_{C_1}\right)+\left(\eta^{im}_{C_2}\right)\right)\right)}, \\ \left(1 - \frac{1}{2}\left((\phi_{C_1}) + (\phi_{C_2})\right)\right).e^{i2\pi\left(1-\frac{1}{2}\left(\left(\phi^{+im}_{C_1}\right)+\left(\phi^{+im}_{C_2}\right)\right)\right)}, \\ \left(\frac{1}{2}\left((r_{C_1}) + (r_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(r^{im}_{C_1}\right)+\left(r^{im}_{C_2}\right)\right)\right)} \end{array}\right);$$

$$\text{(ii) } P_1 \oplus^2 P_2 = \begin{pmatrix} \left(\frac{1}{2}\left((\eta_{C_1}) + (\eta_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{im}\right)+\left(\eta_{C_2}^{im}\right)\right)\right)}, \\ \left(1-\frac{1}{2}\left((\phi_{C_1}) + (\phi_{C_2})\right)\right) e^{i2\pi\left(1-\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)+\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left(1-\frac{1}{2}\left((r_{C_1}) + (r_{C_2})\right)\right) e^{i2\pi\left(1-\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)+\left(r_{C_2}^{+im}\right)\right)\right)} \end{pmatrix}$$

$$\text{(iii) } P_1 \otimes^1 P_2 = \begin{pmatrix} \left(1-\frac{1}{2}\left((\eta_{C_1}) + (\eta_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{+im}\right)+\left(\eta_{C_2}^{+im}\right)\right)\right)}, \\ \left[\left(\frac{1}{2}\left((\phi_{C_1}) + (\phi_{C_2})\right)\right)\right].e^{i2\pi\left(\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)+\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left(1-\frac{1}{2}\left((r_{C_1}) + (r_{C_2})\right)\right) e^{i2\pi\left(\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)+\left(r_{C_2}^{+im}\right)\right)\right)}, \end{pmatrix};$$

$$\text{(iii) } P_1 \otimes^2 P_2 = \begin{pmatrix} \left(1-\frac{1}{2}\left((\eta_{C_1}) + (\eta_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{+im}\right)+\left(\eta_{C_2}^{+im}\right)\right)\right)}, \\ \left[\left(\frac{1}{2}\left((\phi_{C_1}) + (\phi_{C_2})\right)\right)\right].e^{i2\pi\left(\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)+\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left[\left(\frac{1}{2}\left((r_{C_1}) + (r_{C_2})\right)\right)\right].e^{i2\pi\left(\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)+\left(r_{C_2}^{+im}\right)\right)\right)} \end{pmatrix};$$

$$\text{(iii) } \alpha^1 P = \begin{pmatrix} (\alpha)\,\eta_C e^{i2\pi(\alpha)\eta_C^{+im}}, \left(\alpha\left(1-(\phi_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(\phi_C^{+im}\right)\right)\right)}, \\ (\alpha)\,r_C.e^{i2\pi(\alpha)^{\frac{1}{3}}r_C^{+im}} \end{pmatrix}, \textit{ where } 0 \le \alpha \le 1;$$

$$\text{(iii) } \alpha^2 P = \begin{pmatrix} (\alpha)\,\eta_C e^{i2\pi(\alpha)\eta_C^{+im}}, \left(\alpha\left(1-(\phi_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(\phi_C^{+im}\right)\right)\right)}, \\ \left(\alpha\left(1-(r_C)\right)\right)^{\frac{1}{3}}.e^{i2\pi\left(\alpha\left(1-\left(r_C^{+im}\right)\right)\right)} \end{pmatrix}, \textit{ where } 0 \le \alpha \le 1;$$

$$\text{(iv) } P^{\lambda} = \begin{pmatrix} l\,(\eta_C)^{\lambda}.e^{i2\pi r^{\frac{1}{3}}\left(\eta_C^{+im}\right)^{\lambda}}, l\,(\phi_C)^{\lambda}.e^{i2\pi r\left(\phi_C^{+im}\right)^{\lambda}} \\ , l\,(r_C)^{\lambda}.e^{i2\pi r^{\frac{1}{3}}\left(r_C^{+im}\right)^{\lambda}} \end{pmatrix},$$ *where l is the total number of CrC-IFNs that are a part of the procedure*;

$$\text{(v) } P^{\odot 1\alpha} = \begin{pmatrix} \left(\alpha\left(1-(\eta_C)\right)\right)^{\frac{1}{3}}.e^{i2\pi\left(\alpha\left(1-\left(\eta_C^{+im}\right)\right)\right)}, (\alpha)\,\phi_C.e^{i2\pi(\alpha)\phi_C^{+im}}, \\ \left(\alpha\left(1-(r_C)\right)\right)^{\frac{1}{3}}.e^{i2\pi\left(\alpha\left(1-\left(r_C^{+im}\right)\right)\right)} \end{pmatrix},$$

where $0 \le \alpha \le 1$.

$$\text{(vi) } P^{\odot 2\alpha} = \begin{pmatrix} \left(\alpha\left(1-(\eta_C)\right)\right)^{\frac{1}{3}}.e^{i2\pi\left(\alpha\left(1-\left(\eta_C^{+im}\right)\right)\right)}, (\alpha)\,\phi_C.e^{i2\pi(\alpha)\phi_C^{+im}}, \\ (\alpha)\,r_C.e^{i2\pi(\alpha)^{\frac{1}{3}}r_C^{+im}} \end{pmatrix},$$

where $0 \le \alpha \le 1$.

Definition 6. *Let* $\left\{P_j = \left(\eta_{C_j} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_j} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_j} e^{i2\pi r_{C_j}^{+im},}\right) : j = 1, 2, ..., m\right\}$ *be the collection of* C_rC*-IF values and let* C_rC*-*$IFWAM : \Omega^m \to \Omega$*. If*

C_rC*-*$IFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\left(\alpha_1^1 P_1\right)^{\lambda} \oplus \left(\alpha_2^1 P_2\right)^{\lambda} \oplus \left(\alpha_3^1 P_3\right)^{\lambda} \oplus ... \oplus \left(\alpha_m^1 P_m\right)^{\lambda}\right)^{\frac{1}{\lambda}}$ *then* C_rC*-*$IFWAM$ *is called a Circular Complex Intuitionistic fuzzy weighted averaging mean operator of dimension* n *, where* Ω *is the set of all* C_rC*-IF values,* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *,* $\Sigma_{r=1}^m \alpha_r = 1$*, where* $r = 1, 2, ..., m$*.*

Theorem 1. *Let* $\left\{P_j = \left(\eta_{C_j} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_j} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_j} e^{i2\pi r_{C_j}^{+im},}\right) : j = 1, 2, ..., m\right\}$ *be the collection of* C_rC*-IF values. Then by using the* C_rC*-*$IFWAM_E$ *operator their aggregated value is also a* C_rC*-IF* *value and*

$$C_r^1 C - IFWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$$
$$= \begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}} .e^{i2\pi\left(\left(\sum\limits_{j=1}^{m}\left(\alpha_r\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum\limits_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum\limits_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum\limits_{j=1}^{m}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}} .e^{i2\pi\left(\left(\sum\limits_{j=1}^{m}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}.$$

$E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma_{r=1}^{n}\alpha_r = 1, r = 1, 2, ..., m.$

Proof. Let

$$\alpha_1^1 P_1 = \begin{pmatrix} \left(\alpha_1\right)\eta_{C_1}.e^{i2\pi(\alpha_1)\eta_{C_1}^{+im}}, \left(\alpha_1\left(1-\left(\phi_{C_1}\right)\right)\right).e^{i2\pi\left(\alpha_1\left(1-\left(\phi_{C_1}^{+im}\right)\right)\right)}, \\ \left(\alpha_1\right)r_{C_1}.e^{i2\pi(\alpha_1)r_{C_1}^{+im}} \end{pmatrix}$$

and

$$\alpha_2^1 P_2 = \begin{pmatrix} \left(\alpha_2\right)\eta_{C_2}e^{i2\pi(\alpha_2)\eta_{C_2}^{+im}}, \left(\alpha_2\left(1-\left(\phi_{C_2}\right)\right)\right)e^{i2\pi\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left(\alpha_2\right)r_{C_2}e^{i2\pi(\alpha_2)r_{C_2}^{+im}} \end{pmatrix}$$

Then

$$\left(\alpha_1^1 P_1\right)^{\lambda} = \begin{pmatrix} l\left(\left(\alpha_1\right)\eta_{C_1}\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_1\right)\eta_{C_1}^{+im}\right)^{\lambda}}, \\ l\left(\left(\alpha_1\left(1-\left(\phi_{C_1}\right)\right)\right)\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_1\left(1-\left(\phi_{C_1}^{+im}\right)\right)\right)\right)^{\lambda}}, \\ l\left(\left(\alpha_1\right)^{\frac{1}{3}} r_{C_1}\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_1\right)r_{C_1}^{+im}\right)^{\lambda}} \end{pmatrix}$$

and

$$\left(\alpha_2^1 P_2\right)^{\lambda} = \begin{pmatrix} l\left(\left(\alpha_2\right)\eta_{C_2}\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_2\right)\eta_{C_2}^{+im}\right)^{\lambda}}, \\ l\left(\left(\alpha_2\left(1-\left(\phi_{C_2}\right)\right)\right)\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)\right)\right)\right)^{\lambda}}, \\ l\left(\left(\alpha_2\right)^{\frac{1}{3}} r_{C_2}\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_2\right)r_{C_2}^{+im}\right)^{\lambda}} \end{pmatrix}.$$

Now

$$
\begin{aligned}
&\left(\alpha_1^1 P_1\right)^\lambda \oplus \left(\alpha_2^1 P_2\right)^\lambda \\
&= \begin{pmatrix}
\left(\frac{1}{2}\begin{pmatrix} \left(2\left((\alpha_1)\,\eta_{C_1}\right)^\lambda\right) \\ +\left(2\left((\alpha_2)^{\frac{1}{3}}\,\eta_{C_2}\right)^\lambda\right)\end{pmatrix}\right).e^{i2\pi\left(\frac{1}{2}\begin{pmatrix} \left(2\left((\alpha_1)\,\eta_{C_1}^{+im}\right)^\lambda\right) \\ +\left(2\left((\alpha_2)^{\frac{1}{3}}\,\eta_{C_2}^{+im}\right)^\lambda\right)\end{pmatrix}\right)}, \\
\left(1-\frac{1}{2}\begin{pmatrix} \left(2\left((\alpha_1\left(1-(\phi_{C_1})\right))\right)^\lambda\right) \\ +\left(2\left((\alpha_2\left(1-(\phi_{C_2})\right))\right)^\lambda\right)\end{pmatrix}\right). \\
e^{i2\pi\left(1-\frac{1}{2}\begin{pmatrix} \left(2\left((\alpha_1\left(1-(\phi_{C_1}^{+im})\right))\right)^\lambda\right) \\ +\left(2\left((\alpha_2\left(1-(\phi_{C_2}^{+im})\right))\right)^\lambda\right)\end{pmatrix}\right)}, \\
\left(\frac{1}{2}\begin{pmatrix} \left(2\left((\alpha_1)\,\eta_{C_1}\right)^\lambda\right) \\ +\left(2\left((\alpha_2)^{\frac{1}{3}}\,\eta_{C_2}\right)^\lambda\right)\end{pmatrix}\right).e^{i2\pi\left[\left(\frac{1}{2}\begin{pmatrix} \left(2\left((\alpha_1)\,\eta_{C_1}^{+im}\right)^\lambda\right) \\ +\left(2\left((\alpha_2)^{\frac{1}{3}}\,\eta_{C_2}^{+im}\right)^\lambda\right)\end{pmatrix}\right)\right]}
\end{pmatrix}
\end{aligned}
$$

$$
= \begin{pmatrix}
\left(\frac{1}{2}\begin{pmatrix} 2\left(\left((\alpha_1)\,\eta_{C_1}\right)^\lambda\right) \\ +2\left(\left((\alpha_2)\,\eta_{C_2}\right)^\lambda\right)\end{pmatrix}\right).e^{i2\pi\left(\frac{1}{2}\begin{pmatrix} 2\left(\left((\alpha_1)\,\eta_{C_1}^{+im}\right)^\lambda\right) \\ +2\left(\left((\alpha_2)\,\eta_{C_2}^{+im}\right)^\lambda\right)\end{pmatrix}\right)}, \\
\left(1-\frac{1}{2}\begin{pmatrix} 2\left(\left((\alpha_1\left(1-(\phi_{C_1})\right))\right)^\lambda\right) \\ +2\left(\left((\alpha_2\left(1-(\phi_{C_2})\right))\right)^\lambda\right)\end{pmatrix}\right). \\
e^{i2\pi\left(1-\frac{1}{2}\begin{pmatrix} 2\left(\left((\alpha_1\left(1-(\phi_{C_1}^{+im})\right))\right)^\lambda\right) \\ +2\left(\left((\alpha_2\left(1-(\phi_{C_2}^{+im})\right))\right)^\lambda\right)\end{pmatrix}\right)}, \\
\left(\frac{1}{2}\begin{pmatrix} 2\left(\left((\alpha_1)\,r_{C_1}\right)^\lambda\right) \\ +2\left(\left((\alpha_2)\,r_{C_2}\right)^\lambda\right)\end{pmatrix}\right).e^{i2\pi\left(\frac{1}{2}\begin{pmatrix} 2\left(\left((\alpha_1)\,r_{C_1}^{+im}\right)^\lambda\right) \\ +2\left(\left((\alpha_2)\,r_{C_2}^{+im}\right)^\lambda\right)\end{pmatrix}\right)}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\begin{pmatrix} \left(\left((\alpha_1)\,\eta_{C_1}^{+}\right)^\lambda\right)+ \\ \left(\left((\alpha_2)\,\eta_{C_2}^{+}\right)^\lambda\right)\end{pmatrix}.e^{i2\pi\left[\begin{pmatrix} \left(\left((\alpha_1)\,\eta_{C_1}^{-im}\right)^\lambda\right)+ \\ \left(\left((\alpha_2)\,\eta_{C_2}^{+im}\right)^\lambda\right)\end{pmatrix}\right]}, \\
\left(1-\begin{pmatrix} \left(\alpha_1\left(1-\left(\phi_{C_1}(u)\right)\right)\right)^\lambda \\ +\left(\alpha_2\left(1-(\phi_{C_2})\right)\right)^\lambda\end{pmatrix}\right).e^{i2\pi\left(1-\begin{pmatrix} \left(\alpha_1\left(1-(\phi_{C_1}^{+im})\right)\right)^\lambda \\ +\left(\alpha_2\left(1-(\phi_{C_2}^{+im})\right)\right)^\lambda\end{pmatrix}\right)} \\
\begin{pmatrix} \left(\left((\alpha_1)\,r_{C_1}^{+}\right)^\lambda\right)+ \\ \left(\left((\alpha_2)\,r_{C_2}^{+}\right)^\lambda\right)\end{pmatrix}.e^{i2\pi\left[\begin{pmatrix} \left(\left((\alpha_1)\,r_{C_1}^{+im}\right)^\lambda\right)+ \\ \left(\left((\alpha_2)\,r_{C_2}^{+im}\right)^\lambda\right)\end{pmatrix}\right]}
\end{pmatrix}
$$

$$= \begin{pmatrix} \left(\sum_{j=1}^{2}\left(\alpha_j\left(\eta_{C_j}^{+}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{2}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)}, \\ \left(1-\sum_{j=1}^{2}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{3}}.e^{i2\pi\left(1-\sum_{j=1}^{2}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{3}}}, \\ \left(\sum_{j=1}^{2}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{2}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)} \end{pmatrix}.$$

Let suppose that the result is true for $j = k$.

$$\left(\left(\alpha_1^1 P_1\right)^{\lambda} \oplus \left(\alpha_2^1 P_2\right)^{\lambda} \oplus \left(\alpha_3^1 P_3\right)^{\lambda} \oplus ... \oplus \left(\alpha_k^1 P_k\right)^{\lambda}\right)$$
$$= \begin{pmatrix} \left(\sum_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)}, \\ \left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{3}}.e^{i2\pi\left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{3}}}, \\ \left(\sum_{j=1}^{k}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)} \end{pmatrix}.$$

We shows that the result is true for $j = k + 1$. So

$$\left(\left(\alpha_1 P_1\right)^{\lambda} \oplus \left(\alpha_2 P_2\right)^{\lambda} \oplus \left(\alpha_3 P_3\right)^{\lambda} \oplus ... \oplus \left(\alpha_k P_k\right)^{\lambda}\right) \oplus \left(\alpha_{k+1} P_{k+1}\right)^{\lambda}$$
$$= \begin{pmatrix} \left(\sum_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)}, \\ \left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{3}}.e^{i2\pi\left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{3}}}, \\ \left(\sum_{j=1}^{k}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)} \end{pmatrix} \oplus$$
$$\left(\begin{pmatrix} l\left(\left(\alpha_{k+1}\right)\eta_{C_{k+1}}\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_{k+1}\right)\eta_{C_{k+1}}^{+im}\right)^{\lambda}}, \\ l\left(\left(\alpha_{k+1}\left(1-\left(\phi_{C_{k+1}}\right)\right)\right)\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_{k+1}\left(1-\left(\phi_{C_{k+1}}^{+im}\right)\right)\right)\right)^{\lambda}}, \\ l\left(\left(\alpha_{k+1}\right)^{\frac{1}{3}} r_{C_{k+1}}\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_1\right) r_{C_{k+1}}^{+im}\right)^{\lambda}} \end{pmatrix}\right)$$

$$= \begin{pmatrix} \left(\sum_{j=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}^{+}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)}, \\ \left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{3}}.e^{i2\pi\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{3}}}, \\ \left(\sum_{j=1}^{k+1}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)} \end{pmatrix}.$$

Thus

$$\left(\left((\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \oplus (\alpha_3 P_3)^\lambda \oplus ... \oplus (\alpha_k P_k)^\lambda\right) \oplus (\alpha_{k+1} P_{k+1})^\lambda\right)^{\frac{1}{\lambda}}$$

$$= \begin{pmatrix} \left(\left(\sum_{j=1}^{k+1} \left(\alpha_j \left(\eta_{C_j}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(\sum_{r=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{k+1} \left(\alpha_j \left(1-\left(\phi_{C_j}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum_{j=1}^{k+1} \left(\alpha_j \left(r_{C_j}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(\sum_{r=1}^{k+1}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}.$$

□

Theorem 2. *Let* $\left\{P_j = \left(\eta_{C_j} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_j} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_j} e^{i2\pi r_{C_j}^{+im},}\right) : j = 1, 2, ..., m\right\}$ *be the collection of* $C_r CT$*-SPF values values. Then the* $C_r^2 C$*-*$IFWAM_E$ *operator is defined by*

$$C_r^2 C - FPFWAM_E \left(P_1, P_2, P_3, ..., P_m\right)$$

$$= \begin{pmatrix} \left(\left(\sum_{r=1}^{m} \left(\alpha_j \left(\eta_{C_j}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(\sum_{j=1}^{m}\left(\alpha_r\left(\eta_{C_j}^{+im}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{m} \left(\alpha_j \left(1-\left(\phi_{C_j}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{m} \left(\alpha_j \left(1-\left(r_{C_j}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_j\left(1-\left(r_{C_j}^{+im}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}.$$

$E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^n \alpha_r = 1, r = 1, 2, ..., m.$

Proof. Similar to the proof of Theorem 1. □

Theorem 3. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_r C$*-*IF *values. Let* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$*. If* $\left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1} e^{i2\pi r_{C_1}^{+im},}\right) =$

Proposition 1. $\left(\eta_{C_2} e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2} e^{i2\pi r_{C_2}^{+im},}\right)$
$= \left(\eta_{C_m} e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{C_m}^{+im}}, r_{C_m} e^{i2\pi r_{C_m}^{+im},}\right)$
$= \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C e^{i2\pi r_C^{+im},}\right)$ *and* $\lambda = 1$,
then $C_r^1 C$*-*$IFWAM_E \left(P_1, P_2, P_3, ..., P_m\right)$
$= \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C e^{i2\pi r_C^{+im},}\right).$

Proof. Let $\left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im},}\right)$,
$= \left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}}, \phi c_2 e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2}e^{i2\pi r_{C_2}^{+im},}\right)$
$= \left(\eta_{C_m}e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m}e^{i2\pi\phi_{C_m}^{+im}}, r_{C_m}e^{i2\pi r_{C_m}^{+im},}\right)$
$= \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im},}\right)$ and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$, where $r = 1, 2, ..., m$. Based on Definition 6, we get

$$C_rC - IFWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$$

$$= \begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}} e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_C\right)\right)\right).e^{i2\pi\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_C^{+im}\right)\right)\right)}, \\ \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_C\right)\right)\right)\right) e^{i2\pi\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_C^{+im}\right)\right)\right)\right)}, \\ \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_C\right)\right)\right).e^{i2\pi\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_C^{+im}\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\left(\eta_C\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right).e^{i2\pi\left(\left(\eta_C^{+im}\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right)}, \\ \left(1-\left(1-\left(\phi_C\right)\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right)^{\frac{1}{3}}.e^{i2\pi\left(1-\left(1-\left(\phi_C^{+im}\right)\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right)}, \\ \left(\left(r_C\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right).e^{i2\pi\left(\left(r_C^{+im}\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\left(\eta_C\right)\right)e^{i2\pi\left(\left(\eta_C^{+im}\right)\right)}, \\ \left(1-\left(1-\left(\phi_C\right)\right)\right)^{\frac{1}{3}}.e^{i2\pi\left(1-\left(1-\left(\phi_C^{+im}\right)\right)\right)}, \\ \left(\left(r_C\right)\right)e^{i2\pi\left(\left(r_C^{+im}\right)\right)} \end{pmatrix}$$

$$= \left(\eta_C.e^{i2\pi\eta_C^{+im}}, \phi_C.e^{i2\pi\phi_C^{+im}}, r_C.e^{i2\pi r_C^{+im}}\right).$$

□

Theorem 4. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* C_rC-IF *values. Let* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r = 1$. *If* $\left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im},}\right)$

Proposition 2. $= \left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}}, \phi c_2 e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2}e^{i2\pi r_{C_2}^{+im},}\right)$

$= \left(\eta_{C_m} e^{i2\pi\eta^{+im}_{C_m}}, \phi_{C_m} e^{i2\pi\phi^{+im}_{C_m}}, r_{C_m} e^{i2\pi r^{+im}_{C_m}},\right)$
$= \left(\eta_{C} e^{i2\pi\eta^{+im}_{C}}, \phi_{C} e^{i2\pi\phi^{+im}_{C}}, r_{C} e^{i2\pi r^{+im}_{C}},\right)$ *and* $\lambda = 1$, *then*
$C^2_r C\text{-}IFWAM_E (P_1, P_2, P_3, ..., P_m)$
$= \left(\eta_{C} e^{i2\pi\eta^{+im}_{C}}, \phi_{C} e^{i2\pi\phi^{+im}_{C}}, r_{C} e^{i2\pi r^{+im}_{C}},\right).$

Proof. Similar to the proof of Theorem 3 □

Theorem 5. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_r C\text{-}IF$ *values and* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$, *where* $r = 1, 2, ..., m$. *Let*

$$P^- = \left(\begin{array}{c} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} \eta^{+im}_{C_r}}, \max\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi^{+im}_{C_r}}, \\ , \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} r^{+im}_{C_r}} \end{array}\right)$$

$$P^+ = \left(\begin{array}{c} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \eta^{+im}_{C_r}}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi^{+im}_{C_r}}, \\ , \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} r^{+im}_{C_r}}, \end{array}\right)$$, *where and* $\lambda = 1$. *Then* $P^- \le C^1_r C\text{-}FFWAM_E (P_1, P_2, P_3, ..., P_m) \le P^+$.

Proof. Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ be the collection of $C_r C\text{-}IF$ values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r, where $r = 1, 2, ..., m$ with $\alpha_r \in [0, 1]$ and $\Sigma^m_{r=1}\alpha_r = 1$. Let

$$P^- = \left(\begin{array}{c} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} \eta^{+im}_{C_r}}, \max\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi^{+im}_{C_r}}, \\ , \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} r^{+im}_{C_r}} \end{array}\right)$$

$$P^+ = \left(\begin{array}{c} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \eta^{+im}_{C_r}}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi^{+im}_{C_r}}, \\ , \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} r^{+im}_{C_r}}, \end{array}\right)$$

where and $\lambda = 1$, we get $(\eta_{C_t}) \le \left(\max\limits_{1\le r\le m} \eta_{C_r}\right)$ where $t = 1, 2, ..., m$. Then we have $\alpha_r (\eta_{C_t}) \le \alpha_r \left(\max\limits_{1\le r\le m} \eta_{C_r}\right)$. This implies that $\sum\limits_{r=1}^m \alpha_r (\eta_{C_t}) \le \sum\limits_{r=1}^m \alpha_r \left(\max\limits_{1\le r\le m} \eta_{C_r}\right)$ then we have, $\sum\limits_{r=1}^m \alpha_r (\eta_{C_t}) \le \left(\max\limits_{1\le r\le m} \eta_{C_r}\right) \sum\limits_{r=1}^m \alpha_r$ This implies that $\sum\limits_{r=1}^m \alpha_r (\eta_{C_t}) \le \left(\max\limits_{1\le r\le m} \eta_{C_r}\right)$ and this implies that $\left(\sum\limits_{r=1}^m \alpha_r (\eta_{C_t})\right) \le \max\limits_{1\le r\le m} \eta_{C_r}$. Hence

$\left(\sum\limits_{r=1}^m \alpha_r (\eta_{C_t})\right) \le \max\limits_{1\le r\le m} \eta_{C_r}$. Also
$\left(\sum\limits_{r=1}^m \alpha_r \left(\eta^{im}_{C_t}\right)\right) \le \max\limits_{1\le r\le m} \eta^{im}_{C_r}$. This implies that
$e^{i2\pi\left(\sum\limits_{r=1}^m \alpha_r\left(\eta^{im}_{C_t}\right)\right)} \le e^{i2\pi \max\limits_{1\le r\le m} \eta^{im}_{C_r}}$. Thus

$$\left(\sum_{r=1}^m \alpha_r \left(\eta^{im}_{C_t}\right)\right).e^{i2\pi\left(\sum\limits_{r=1}^m \alpha_r\left(\eta^{im}_{C_t}\right)\right)}$$
$$\le \max_{1\le r\le m} \eta^{im}_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \eta^{im}_{C_r}}.$$

On the other hand, we get $(\phi_{C_r}) \geq \left(\min_{1\leq r\leq m} \phi_C\right)$. This implies that $-(\phi_{C_r}) \leq -\left(\min_{1\leq r\leq m} \phi_C\right)$ and , also $1-(\phi_{C_r}) \leq 1-\left(\min_{1\leq r\leq m} \phi_C\right)$, where, $r = 1,2,...,m$. Then we have $\alpha_r(1-(\phi_{C_r}(u))) \leq \alpha_r(1-(\phi_C(u)))\, \alpha_r(1-(\phi_{C_r})) \leq \alpha_r\left(1-\left(\min_{1\leq r\leq m} \phi_C\right)\right)$ then we get,

$$\begin{aligned}
\alpha_r(1-(\phi_{C_r})) &\leq \alpha_r\left(1-\left(\min_{1\leq r\leq m} \phi_C\right)\right) \\
&\Rightarrow \sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r})) \leq \sum_{r=1}^{m} \alpha_r\left(1-\left(\min_{1\leq r\leq m} \phi_C\right)\right) \\
&\Rightarrow \sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r})) \leq \left(1-\left(\min_{1\leq r\leq m} \phi_C\right)\right)\sum_{r=1}^{m} \alpha_r \\
&\Rightarrow -\sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r})) \geq -\left(1-\left(\min_{1\leq r\leq m} \phi_C\right)\right) \\
&\Rightarrow 1-\sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r})) \geq 1-\left(1-\left(\min_{1\leq r\leq m} \phi_C\right)\right) \\
&\Rightarrow 1-\sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r})) \geq \left(\min_{1\leq r\leq m} \phi_C\right) \\
&\Rightarrow \left(1-\sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r}))\right) \geq \min_{1\leq r\leq m} \phi_C.
\end{aligned}$$

Also $\left(1-\sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r}))\right) \geq \min_{1\leq r\leq m} \phi_C$. This implies that
$\left(1-\sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r}))\right) \geq \min_{1\leq r\leq m} \phi_C$. Also
$\left(1-\sum_{r=1}^{m} \alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right) \geq \min_{1\leq r\leq m} \phi_C^{+im}$. This implies that
$e^{i2\pi\left(1-\sum_{r=1}^{m} \alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)\right)\right)} \geq e^{i2\pi \min_{1\leq r\leq m} \phi_C^{+im}}$. Thus

$$\begin{aligned}
&\left(1-\sum_{r=1}^{m} \alpha_r(1-(\phi_{C_r}))\right).e^{i2\pi\left(1-\sum_{r=1}^{m} \alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)\right)\right)} \\
\geq\ & \min_{1\leq r\leq m} \phi_C.e^{i2\pi \min_{1\leq r\leq m} \phi_C^{-im}}.
\end{aligned}$$

On the other hand, $(r_{C_t}) \leq \left(\max_{1\leq r\leq m} r_{C_r}\right)$ where $t = 1,2,...,m$. Then we have $r_r(\eta_{C_t}) \leq r_r\left(\max_{1\leq r\leq m} \eta_{C_r}\right)$. This implies that $\sum_{r=1}^{m} \alpha_r(r_{C_t}) \leq \sum_{r=1}^{m} \alpha_r\left(\max_{1\leq r\leq m} r_{C_r}\right)$ then we have, $\sum_{r=1}^{m} \alpha_r(r_{C_t}) \leq \left(\max_{1\leq r\leq m} r_{C_r}\right)\sum_{r=1}^{m} \alpha_r$ This implies that $\sum_{r=1}^{m} \alpha_r(r_{C_t}) \leq \left(\max_{1\leq r\leq m} r_{C_r}\right)$ and this implies that $\left(\sum_{r=1}^{m} \alpha_r(r_{C_t})\right)^{\frac{1}{3}} \leq \max_{1\leq r\leq m} r_{C_r}$. Hence
$\left(\sum_{r=1}^{m} \alpha_r(r_{C_t})\right) \leq \max_{1\leq r\leq m} r_{C_r}$. Also

$\left(\sum_{r=1}^{m}\alpha_r\left(r_{C_t}^{im}\right)\right)\leq\max_{1\leq r\leq m}r_{C_r}^{im}$. This implies that
$e^{i2\pi\left(\sum_{r=1}^{m}\alpha_r\left(r_{C_t}^{im}\right)\right)}\leq e^{i2\pi\max_{1\leq r\leq m}r_{C_r}^{im}}$. Thus

$$\begin{aligned}&\left(\sum_{r=1}^{m}\alpha_r\left(r_{C_t}^{im}\right)\right).e^{i2\pi\left(\sum_{r=1}^{m}\alpha_r\left(r_{C_t}^{im}\right)\right)}\\ \leq\ &\max_{1\leq r\leq m}r_{C_r}^{im}.e^{i2\pi\max_{1\leq r\leq m}r_{C_r}^{im}}.\end{aligned}$$

Based on Definition **??** and Definition 6, we get $CrC\text{-}IFWAM_E\left(P_1,P_2,P_3,...,P_m\right)\leq P^{+}$. Similarly, we can have $C_r^1C\text{-}IFWAM_E\left(P_1,P_2,P_3,...,P_m\right)\geq P^{-}$. Finally we get,

$$P^{+}\geq C_r^1C-IFWAM_E\left(P_1,P_2,P_3,...,P_m\right)\geq P^{-}.$$

□

Theorem 6. *Let* $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ *be the collection of* $C_r^2C\text{-}IF$ *values and* $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^m\alpha_r=1$*, where* $r=1,2,...,m$*. Let*
$P^{-}=\left(\min_{1\leq r\leq m}\eta_{C_r}.e^{i2\pi\min_{1\leq r\leq m}\eta_{C_r}^{+im}},\max_{1\leq r\leq m}\phi_{C_r}.e^{i2\pi\max_{1\leq r\leq m}\phi_{C_r}^{+im}},\max_{1\leq r\leq m}r_{C_r}.e^{i2\pi\min_{1\leq r\leq m}r_{C_r}^{+im}}\right)$
$P^{+}=\left(\begin{array}{c}\max_{1\leq r\leq m}\eta_{C_r}.e^{i2\pi\max_{1\leq r\leq m}\eta_{C_r}^{+im}},\min_{1\leq r\leq m}\phi_{C_r}.e^{i2\pi\max_{1\leq r\leq m}\phi_{C_r}^{+im}},\\ ,\min_{1\leq r\leq m}r_{C_r}.e^{i2\pi\max_{1\leq r\leq m}r_{C_r}^{+im}},\end{array}\right)$*, where and* $\lambda=1$*. Then* $P^{-}\leq C_r^2C\text{-}FFWAM_E\left(P_1,P_2,P_3,...,P_m\right)\leq P^{+}$.

Proof. Similar to the proof of Theorem 5. □

Theorem 7. *Let* $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ *and*
$\left\{P_r^{*}=\left(\eta_{C_r}^{*}e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^{*}e^{i2\pi\phi_{C_r}^{+im*}},r_{C_r}^{*}e^{i2\pi r_{C_r}^{+im*}}\right):r=1,2,...,m\right\}$ *are two collections of* $C_rC\text{-}IF$ *values. If* $\eta_{C_r}\leq\eta_{C_r}^{*}$, $\eta_{C_r}^{im}\leq\eta_{C_r}^{im*}$, $\phi_{C_r}\geq\phi_{C_r}^{*}$, $\phi_{C_r}^{im}\geq\phi_{C_r}^{im*}$, $r_{C_r}\leq r_{C_r}^{*}$, *and* $r_{C_r}^{im}\leq r_{C_r}^{im*}$ *where* $r=1,2,...,m$*, then* $C_r^1C\text{-}IFWAM_E\left(P_1,P_2,P_3,...,P_m\right)\leq C_r^1C\text{-}IFWAM_E\left(P_1^{*},P_2^{*},P_3^{*},...,P_m^{*}\right)$.

Proof. Given that $P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right)$ and
$P_r^{*}=\left(\eta_{C_r}^{*}e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^{*}e^{i2\pi\phi_{C_r}^{+im*}},r_{C_r}^{*}e^{i2\pi r_{C_r}^{+im*}}\right)$ are two collections of $C_rC\text{-}IF$ values, where $r=1,2,...,m$. If $\eta_{C_r}\leq\eta_{C_r}^{*}$, $\eta_{C_r}^{im}\leq\eta_{C_r}^{im*}$, $\phi_{C_r}\geq\phi_{C_r}^{*}$, $\phi_{C_r}^{im}\geq\phi_{C_r}^{im*}$, $r_{C_r}\leq r_{C_r}^{*}$, and $r_{C_r}^{im}\leq r_{C_r}^{im*}$ then $\left(\eta_{C_r}\right)\leq\left(\eta_{C_r}^{*}\right)$. As we have

$$\begin{aligned}\left(\eta_{C_r}\right)&\leq\left(\eta_{C_r}^{*}\right)\\ &\Rightarrow\alpha_r\left(\eta_{C_r}\right)\leq\alpha_r\left(\eta_{C_r}^{*}\right)\\ &\Rightarrow\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\leq\left(\alpha_r\left(\eta_{C_r}^{*}\right)\right)^{\lambda}\\ &\Rightarrow\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\leq\left(\alpha_r\left(\eta_{C_r}^{*}\right)\right)^{\lambda}\end{aligned}$$

$$\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).$$

Thus

$$\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right). \qquad \text{(i)}$$

Further,

$$\begin{aligned}
\left(\eta_{C_r}^{im}(u)\right) &\leq \left(\eta_{C_r}^{im*}(u)\right) \leq \\
&\Rightarrow \alpha_r\left(\eta_{C_r}^{im}(u)\right) \leq \alpha_r\left(\eta_{C_r}^{im*}(u)\right) \\
&\Rightarrow \left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)\right)^{\lambda} \leq \left(\alpha_r\left(\eta_{C_r}^{im*}(u)\right)\right)^{\lambda} \\
&\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)\right)^{\lambda} \leq \left(\alpha_r\left(\eta_{C_r}^{im*}(u)\right)\right)^{\lambda}
\end{aligned}$$

$$\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).$$

Hence

$$e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \leq e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}}. \qquad \text{(ii)}$$

From (i) and (ii)

$$\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}$$

$$\leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}\right]}. \qquad (A)$$

Now

$$\begin{aligned}
\left(\phi_{C_r}\right) &\geq \left(\phi_{C_r}^{*}\right) \\
&\Rightarrow -\left(\phi_{C_r}\right) \leq -\left(\phi_{C_r}^{*}\right) \\
&\Rightarrow \left(1-\left(\phi_{C_r}\right)\right) \leq \left(1-\left(\phi_{C_r}^{*}\right)\right) \\
&\Rightarrow \alpha_r\left(1-\left(\phi_{C_r}\right)\right) \leq \alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right) \\
&\Rightarrow \left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda} \leq \left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right)\right)^{\lambda}.
\end{aligned}$$

Then we have

$$
\begin{aligned}
& \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda} \\
\leq\; & \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right)\right)^{\lambda} \\
\Rightarrow\; & -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda} \geq -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right)\right)^{\lambda} \\
\Rightarrow\; & 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda} \geq 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right)\right)^{\lambda} \\
\Rightarrow\; & \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \geq \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
\Rightarrow\; & \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right). \qquad \text{(iii)}
\end{aligned}
$$

Also

$$
\begin{aligned}
& \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda} \\
\leq\; & \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)\right)\right)^{\lambda} \\
\Rightarrow\; & -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda} \geq -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)\right)\right)^{\lambda} \\
\Rightarrow\; & 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda} \geq 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)\right)\right)^{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\; & \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \geq \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
\Rightarrow\; & \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).
\end{aligned}
$$

This implies that

$$
e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \geq e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}. \qquad \text{(iv)}
$$

From (iii) and (iv) , we get

$$
\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1\left(\phi_{C_r}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}
$$

$$\geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{+im*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}. \qquad (B).$$

Now

$$\begin{aligned}
\left(r_{C_r}\right) &\leq \left(r^*_{C_r}\right) \\
&\Rightarrow \alpha_r\left(r_{C_r}\right) \leq \alpha_r\left(r^*_{C_r}\right) \\
&\Rightarrow \left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda} \leq \left(\alpha_r\left(r^*_{C_r}\right)\right)^{\lambda} \\
&\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda} \leq \left(\alpha_r\left(r^*_{C_r}\right)\right)^{\lambda}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(r^*_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
&\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^*_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).
\end{aligned}$$

Thus

$$\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}} \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^*_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right). \qquad \text{(vii)}$$

Further,

$$\begin{aligned}
\left(r^{im}_{C_r}(u)\right) &\leq \left(r^{im*}_{C_r}(u)\right) \leq \\
&\Rightarrow \alpha_r\left(r^{im}_{C_r}(u)\right) \leq \alpha_r\left(r^{im*}_{C_r}(u)\right) \\
&\Rightarrow \left(\alpha_r\left(r^{im}_{C_r}(u)\right)\right)^{\lambda} \leq \left(\alpha_r\left(r^{im*}_{C_r}(u)\right)\right)^{\lambda} \\
&\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(r^{im}_{C_r}(u)\right)\right)^{\lambda} \leq \left(\alpha_r\left(r^{im*}_{C_r}(u)\right)\right)^{\lambda}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(r^{im}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(r^{im*}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
&\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^{im}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^{im*}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).
\end{aligned}$$

Hence

$$e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \leq e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}. \qquad \text{(viii)}$$

From (i) and (ii)

$$\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}.e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}$$

$$\leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^*_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^{+im*}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}. \qquad (D)$$

Thus, from (A) ,(B) ,(C) and (D) , we have

$$\begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{+im}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1\left(\phi^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}.e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^{+im}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}$$

$$\leq \begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^*_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{+im*}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi^{+im*}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^*_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}.e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^{+im*}_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \end{pmatrix}.$$

Hence $CrC\text{-}IFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq CrC\text{-}FFWAM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right).$ ☐

Theorem 8. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *and* $\left\{P_r^* = \left(\eta^*_{C_r}e^{i2\pi\eta^{+im*}_{C_r}}, \phi^*_{C_r}e^{i2\pi\phi^{+im*}_{C_r}}, r^*_{C_r}e^{i2\pi r^{+im*}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* $C_rC\text{-}IF$ *values. If* $\eta_{C_r} \leq \eta^*_{C_r}$, $\eta^{im}_{C_r} \leq \eta^{im*}_{C_r}$, $\phi_{C_r} \geq \phi^*_{C_r}$, $\phi^{im}_{C_r} \geq \phi^{im*}_{C_r}$, $r_{C_r} \geq r^*_{C_r}$, *and* $r^{im}_{C_r} \geq r^{im*}_{C_r}$ *where* $r = 1, 2, ..., m$, *then* $C_r^2C\text{-}IFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq C_r^2C\text{-}IFWAM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right).$

Proof. Similar to the proof of Theorem 7. ☐

Definition 7. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}IF$ *values and let* $C_r^1C\text{-}IFOWAM : \Omega^m \to \Omega$, *if*

$C_r^1C\text{-}IFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) =$

$\left(\left(\alpha_1P_{\delta(1)}\right)^{\lambda} \oplus \left(\alpha_2P_{\delta(1)}\right)^{\lambda} \oplus \left(\alpha_3P_{\delta(3)}\right)^{\lambda} \oplus ... \oplus \left(\alpha_mP_{\delta(m)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$ *then* $C_r^1C\text{-}IFOWAM$ *is called a Circular Complex Intuitionistic fuzzy ordered weighted averaging mean operator of dimension* n, *where* $\left(\delta\left(1\right), \delta\left(2\right), ..., \delta\left(m\right)\right)$ *is a permutation of* $(1, 2, ..., m)$ *such that* $P_{\delta(r-1)} \geq P_{\delta(r)}$ *for all* r, Ω *is the set of all* $C_rC\text{-}IF$ *values,* $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m\alpha_r = 1$, *where* $r = 1, 2, ..., m$.

Theorem 9. *Let* $P_r = \left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right)$ *be the collection of* CrC*-IF values, where* $r = 1, 2, ..., m$. *Then by using the* $C_rC\text{-}IFOWAM_E$ *operator their aggregated value is also a*

C_r^1C *-IF value and*

$$C_r^1C - IFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$$
$$= \left(\begin{array}{c} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_{\delta(r)}}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_{\delta(r)}}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_{\delta(r)}}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_{\delta(r)}}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_{\delta(r)}}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_{\delta(r)}}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \end{array}\right)$$

where $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^n \alpha_r = 1$, where .

Proof. Similar to the proof of Theorem 1. □

Theorem 10. *Let $P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right)$ be the collection of CrC-IF values, where $r = 1, 2, ..., m$. Then C_r^2C-$IFOWAM_E$ operator is defined by*

$$C_r^2C - FPFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$$
$$= \left(\begin{array}{c} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_{\delta(r)}}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_{\delta(r)}}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_{\delta(r)}}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_{\delta(r)}}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(r_{C_{\delta(r)}}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(r_{C_{\delta(r)}}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, , \end{array}\right)$$

where $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^n \alpha_r = 1$, where .

Proof. Similar to the proof of Theorem 1. □

Theorem 11. *Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-IF values. Let $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^m \alpha_r = 1$. If $\left(\eta_{C_1}.e^{i2\pi\left[\eta_{C_1}^{+im}\right]}, \phi_{C_1} e^{i2\pi\left[\phi_{C_1}^{+im}\right]}, r_{C_1}.e^{i2\pi\left[\eta_{C_1}^{+im}\right]}\right)$*
$= \left(\eta_{C_2}.e^{i2\pi\left[\eta_{C_2}^{+im}\right]}, \phi_{C_2} e^{i2\pi\left[\phi_{C_2}^{+im}\right]}, r_{C_2}.e^{i2\pi\left[\eta_{C_2}^{+im}\right]}\right) = ...$
$= \left(\eta_{C_m}.e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{C_m}^{+im}}, r_{C_m}.e^{i2\pi\eta_{C_m}^{+im}}\right)$
$= \left(\eta_{C}.e^{i2\pi\eta_{C}^{+im}}, \phi_{C} e^{i2\pi\phi_{C}^{+im}}, r_{C}.e^{i2\pi\eta_{C}^{+im}}\right)$ *and $\lambda = 1$, then*
C_r^1C-$IFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_{C}.e^{i2\pi\eta_{C}^{+im}}, \phi_{C} e^{i2\pi\phi_{C}^{+im}}, r_{C}.e^{i2\pi\eta_{C}^{+im}}\right).$

Proof. Similar to the proof of Theorem 3. □

Theorem 12. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_r C$*-IF values. Let* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$. *If* $\left(\eta_{C_1}.e^{i2\pi\left[\eta^{+im}_{C_1}\right]}, \phi_{C_1} e^{i2\pi\left[\phi^{+im}_{C_1}\right]}, r_{C_1}.e^{i2\pi\left[\eta^{+im}_{C_1}\right]}\right)$
$= \left(\eta_{C_2}.e^{i2\pi\left[\eta^{+im}_{C_2}\right]}, \phi_{C_2} e^{i2\pi\left[\phi^{+im}_{C_2}\right]}, r_{C_2}.e^{i2\pi\left[\eta^{+im}_{C_2}\right]}\right)$
$= \left(\eta_{C_m}.e^{i2\pi\eta^{+im}_{C_m}}, \phi_{C_m} e^{i2\pi\phi^{+im}_{C_m}}, r_{C_m}.e^{i2\pi\eta^{+im}_{C_m}}\right)$
$= \left(\eta_{C}.e^{i2\pi\eta^{+im}_{C}}, \phi_{C} e^{i2\pi\phi^{+im}_{C}}, r_{C}.e^{i2\pi\eta^{+im}_{C}}\right)$ *and* $\lambda = 1$, *then*
$C^2_r C\text{-}IFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_{C}.e^{i2\pi\eta^{+im}_{C}}, \phi_{C} e^{i2\pi\phi^{+im}_{C}}, r_{C}.e^{i2\pi\eta^{+im}_{C}}\right)$.

Proof. Similar to the proof of Theorem 3. □

Theorem 13. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_r C$*-IF values and* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$. *Let*
$P^- = \left(\min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} \eta^{im}_{C_r}}, \max\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi^{im}_{C_r}}, \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} r^{im}_{C_r}}\right)$ *and*
$P^+ = \left(\max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \eta^{im}_{C_r}}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} \phi^{im}_{C_r}}, \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} r^{im}_{C_r}}\right)$, *where and* $\lambda = 1$.
Then $P^- \le C^1_r C\text{-}IFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \le P^+$.

Proof. Similar to the proof of Theorem 5. □

Theorem 14. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_r C$*-IF values and* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$. *Let*
$P^- = \left(\min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} \eta^{im}_{C_r}}, \max\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi^{im}_{C_r}}, \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} r^{im}_{C_r}}\right)$ *and*
$P^+ = \left(\max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \eta^{im}_{C_r}}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} \phi^{im}_{C_r}}, \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} r^{im}_{C_r}}\right)$,
where and $\lambda = 1$. *Then* $P^- \le C^2_r C\text{-}FFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \le P^+$.

Proof. Similar to the proof of Theorem 5. □

Theorem 15. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *and*
$\left\{P^*_r = \left(\eta^*_{C_r} e^{i2\pi\eta^{+im*}_{C_r}}, \phi^*_{C_r} e^{i2\pi\phi^{+im*}_{C_r}}, r^*_{C_r} e^{i2\pi r^{+im*}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* $C_r C$*-IF values. If* $\eta_{C_r} \le \eta^*_{C_r}$, $\eta^{im}_{C_r} \le \eta^{im*}_{C_r}$, $\phi_{C_r} \ge \phi^*_{C_r}$, $\phi^{im}_{C_r} \ge \phi^{im*}_{C_r}$, $r_{C_r} \le r^*_{C_r}$, *and* $r^{im}_{C_r} \le r^{im*}_{C_r}$ *where* $r = 1, 2, ..., m$, *then* $C^1_r C\text{-}IFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \le C^1_r C\text{-}IFOWAM_E\left(P^*_1, P^*_2, P^*_3, ..., P^*_m\right)$

Proof. Similar to the proof of Theorem 7. □

Theorem 16. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *and*
$\left\{P^*_r = \left(\eta^*_{C_r} e^{i2\pi\eta^{+im*}_{C_r}}, \phi^*_{C_r} e^{i2\pi\phi^{+im*}_{C_r}}, r^*_{C_r} e^{i2\pi r^{+im*}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* $C_r C$*-IF values. If* $\eta_{C_r} \le \eta^*_{C_r}$, $\eta^{im}_{C_r} \le \eta^{im*}_{C_r}$, $\phi_{C_r} \ge \phi^*_{C_r}$, $\phi^{im}_{C_r} \ge \phi^{im*}_{C_r}$, , $r_{C_r} \ge r^*_{C_r}$, *and* $r^{im}_{C_r} \ge r^{im*}_{C_r}$ *where* $r = 1, 2, ..., m$, *then* $C^2_r C\text{-}IFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \le C^2_r C\text{-}IFOWAM_E\left(P^*_1, P^*_2, P^*_3, ..., P^*_m\right)$

Proof. Similar to the proof of Theorem 7. □

3. Circular Complex Intuitionistic fuzzy weighted geometric mean aggregation operators

Here, we provide some new geometric aggregation operators, complex interval valued $C_rC\text{-}IF$ weighted geometric mean aggregation operator (C_rC-FPFWGM) and $C_rC\text{-}IF$ ordered weighted geometric mean aggregation operator ($C_rC\text{-}IFOWGM$), using the suggested operations.

Definition 8. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right) : r = 1,2,...,m\right\}$ *be the collection of* $C_rC\text{-}IF$ *values and let* $C_rC\text{-}IFOWGM : \Omega^m \to \Omega$, *if*

$C_rC\text{-}IFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\left(P_1^{\odot\alpha_1}\right)^{\lambda} \otimes \left(P_2^{\odot\alpha_2}\right)^{\lambda} \otimes \left(P_3^{\odot\alpha_3}\right)^{\lambda} \otimes ... \otimes \left(P_m^{\odot\alpha_m}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$ *then* $C_rC\text{-}IFWGM$ *is called a Circular Complex Intuitionistic fuzzy weighted geometric mean operator of dimension* n, *where* Ω *is the set of all* $C_rC\text{-}IF$ *values,* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$, *where* $r = 1,2,...,m$.

Theorem 17. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right) : r = 1,2,...,m\right\}$ *be the collection of* $C_rC\text{-}IF$ *values. Then by using the* $C_rC\text{-}IFWGM_E$ *operator their aggregated value is also a* $C_rC\text{-}IF$ *value and*

$$\begin{aligned} &= C^1_rC - TSFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \\ &= \begin{pmatrix} \left(\left(1 - \sum\limits_{r=1}^{m}\left(\alpha_r\left(1 - \left(\eta_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\phi^{+im}_{C_r}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1 - \sum\limits_{r=1}^{m}\left(\alpha_r\left(1 - \left(r_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(r^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \end{pmatrix}, \end{aligned}$$

where $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma^n_{r=1}\alpha_r = 1$, *where* $r = 1,2,...,m$.

Proof. Similar to the proof of Theorem 1. □

Theorem 18. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right) : r = 1,2,...,m\right\}$ *be the collection of* $C_rC\text{-}IF$ *values. Then by using the* $C_rC\text{-}IFWGM_E$ *operator their aggregated value is also a* $C_rC\text{-}IF$ *value and*

$$\begin{aligned} &= C^2_rC - TSFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \\ &= \begin{pmatrix} \left(\left(1 - \sum\limits_{r=1}^{m}\left(\alpha_r\left(1 - \left(\eta_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\phi^{+im}_{C_r}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(r^{+im}_{C_r}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}\right]} \end{pmatrix}, \end{aligned}$$

where $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{n}\alpha_r = 1$, where $r = 1, 2, ..., m$.

Proof. Similar to the proof of Theorem 1. □

Proposition 3. *Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of $C_rC\text{-}IF$ values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$.*

If $\left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1} e^{i2\pi r_{C_1}^{+im}}\right) =$
$\left(\eta_{C_2} e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2} e^{i2\pi r_{C_2}^{+im}}\right) = ...$
$= \left(\eta_{C_m} e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{C_m}^{+im}}, r_{C_m} e^{i2\pi r_{C_m}^{+im}}\right)$
$= \left(\eta_{C} e^{i2\pi\eta_{C}^{+im}}, \phi_{C} e^{i2\pi\phi_{C}^{+im}}, r_{C} e^{i2\pi r_{C}^{+im}}\right)$ *and $\lambda = 1$, then*
$C_r^1C\text{-}IFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_{C} e^{i2\pi\eta_{C}^{+im}}, \phi_{C} e^{i2\pi\phi_{C}^{+im}}, r_{C} e^{i2\pi r_{C}^{+im}}\right).$

Proof. Similar to the proof of Theorem 3. □

Proposition 4. *Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of $C_rC\text{-}IF$ values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$.*

If $\left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1} e^{i2\pi r_{C_1}^{+im}}\right) =$
$\left(\eta_{C_2} e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2} e^{i2\pi r_{C_2}^{+im}}\right) = ...$
$= \left(\eta_{C_m} e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{C_m}^{+im}}, r_{C_m} e^{i2\pi r_{C_m}^{+im}}\right)$
$= \left(\eta_{C} e^{i2\pi\eta_{C}^{+im}}, \phi_{C} e^{i2\pi\phi_{C}^{+im}}, r_{C} e^{i2\pi r_{C}^{+im}}\right)$ *and $\lambda = 1$, then*
$C_r^2C\text{-}IFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_{C} e^{i2\pi\eta_{C}^{+im}}, \phi_{C} e^{i2\pi\phi_{C}^{+im}}, r_{C} e^{i2\pi r_{C}^{+im}}\right).$

Proof. Similar to the proof of Theorem 3. □

Theorem 19. *Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of $C_rC\text{-}IF$ values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$. Let*

$$P^- = \left(\begin{array}{c} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} \eta_{C_r}^{+im}\right]}, \max\limits_{1\le r\le m} \phi_{C_r}^{-}.e^{i2\pi\left[\max\limits_{1\le r\le m} \phi_{C_r}^{+im}\right]}, \\ \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} r_{C_r}^{+im}\right]} \end{array}\right) \text{ and}$$

$$P^+ = \left(\begin{array}{c} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} \eta_{C_r}^{+im}\right]}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} \phi_{C_r}^{+im}\right]}, \\ \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} r_{C_r}^{+im}\right]} \end{array}\right),$$

where and $\lambda = 1$. Then $P^- \le C_r^1C - IFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \le P^+$.

Proof. Similar to the proof of Theorem 5. □

Theorem 20. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* C_rC-IF *values and* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$. *Let*

$$P^- = \left(\begin{array}{c} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} \eta^{+im}_{C_r}\right]}, \max\limits_{1\le r\le m} \phi^-_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} \phi^{+im}_{C_r}\right]}, \\ \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} r^{+im}_{C_r}\right]} \end{array}\right) \text{ and}$$

$$P^+ = \left(\begin{array}{c} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} \eta^{+im}_{C_r}\right]}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} \phi^{+im}_{C_r}\right]}, \\ \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} r^{+im}_{C_r}\right]} \end{array}\right),$$ *where and* $\lambda = 1$. *Then* $P^- \le C^2_rC - IFWGM_E(P_1, P_2, P_3, ..., P_m) \le P^+$.

Proof. Similar to the proof of Theorem 5. □

Theorem 21. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *and* $\left\{P^*_r = \left(\eta^*_{C_r} e^{i2\pi\eta^{+im*}_{C_r}}, \phi^*_{C_r} e^{i2\pi\phi^{+im*}_{C_r}}, r^*_{C_r} e^{i2\pi r^{+im*}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* C_rC-IF *values.*

If f $\eta_{C_r} \le \eta^*_{C_r}$, $\eta^{im}_{C_r} \le \eta^{im*}_{C_r}$, $\phi_{C_r} \ge \phi^*_{C_r}$, $\phi^{im}_{C_r} \ge \phi^{im*}_{C_r}$, $r_{C_r} \le r^*_{C_r}$, *and* $r^{im}_{C_r} \le r^{im*}_{C_r}$ *, where* $r = 1, 2, ..., m$, *then* C_rC-$FFWGM_E(P_1, P_2, P_3, ..., P_m) \le C^1_rC - IFWGM_E(P^*_1, P^*_2, P^*_3, ..., P^*_m)$.

Proof. Similar to the proof of Theorem 7. □

Theorem 22. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *and* $\left\{P^*_r = \left(\eta^*_{C_r} e^{i2\pi\eta^{+im*}_{C_r}}, \phi^*_{C_r} e^{i2\pi\phi^{+im*}_{C_r}}, r^*_{C_r} e^{i2\pi r^{+im*}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* C_rC-IF *values. If f* $\eta_{C_r} \le \eta^*_{C_r}$, $\eta^{im}_{C_r} \le \eta^{im*}_{C_r}$, $\phi_{C_r} \ge \phi^*_{C_r}$, $\phi^{im}_{C_r} \ge \phi^{im*}_{C_r}$, $r_{C_r} \le r^*_{C_r}$, *and* $r^{im}_{C_r} \le r^{im*}_{C_r}$ *, where* $r = 1, 2, ..., m$, *then* C_rC-$IFWGM_E(P_1, P_2, P_3, ..., P_m) \le C^2_rC - IFWGM_E(P^*_1, P^*_2, P^*_3, ..., P^*_m)$.

Proof. Similar to the proof of Theorem 7. □

Definition 9. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* C_rC-F *values and let* C_rC-$IFOWGM : \Omega^m \to \Omega$, *if*

$$C_rC\text{-}IFWGM_E(P_1, P_2, P_3, ..., P_m) = \left(\left(P^{\odot\alpha_1}_{\delta(1)}\right)^\lambda \otimes \left(P^{\odot\alpha_2}_{\delta(2)}\right)^\lambda \otimes \left(P^{\odot\alpha_3}_{\delta(3)}\right)^\lambda \otimes ... \otimes \left(P^{\odot\alpha_m}_{\delta(r)}\right)^\lambda\right)^{\frac{1}{\lambda}}$$

then C_rC- $IFOWGM_E$ *is called a complex interval valued* q*-rung orthopair fuzzy ordered weighted geometric operator of dimension* n, *where* $(\delta(1), \delta(2), ..., \delta(m))$ *is a permutation of* $(1, 2, ..., m)$ *such that* $P_{\delta(r-1)} \ge P_{\delta(r)}$ *for all* r, Ω *is the set of all* C_rC-IF *values,* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$.

Theorem 23. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* C_rC-IF *values. Then by using the* C_rC-$IFOWGM_E$ *operator their aggregated value is also a* C_rC-

IF value and

$$C_r^1C-IFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)$$
$$=\left(\begin{array}{c}\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta^{-}_{C_{\delta(r)}}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta^{-im}_{C_{\delta(r)}}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]},\\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi^{-}_{C_{\delta(r)}}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi^{-im}_{C_{\delta(r)}}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]},\\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta^{-}_{C_{\delta(r)}}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta^{-im}_{C_{\delta(r)}}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]},\end{array}\right),$$

where $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{n}\alpha_r=1$.

Proof. Similar to the proof of Theorem 1. □

Proposition 5. *Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}},\phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}},r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-IF values. Let $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$.*

If $\left(\eta_{C_1}e^{i2\pi\eta^{+im}_{C1}},\phi_{C_1}e^{i2\pi\phi^{+im}_{C1}},r_{C_1}e^{i2\pi r^{+im}_{C1}}\right)=$
$=\left(\eta_{C_2}e^{i2\pi\eta^{+im}_{C_2}},\phi_{C_2}e^{i2\pi\phi^{+im}_{C_2}},r_{C_2}e^{i2\pi r^{+im}_{C_2}}\right)=...$
$=\left(\eta_{C_m}e^{i2\pi\eta^{+im}_{Cm}},\phi_{C_m}e^{i2\pi\phi^{+im}_{Cm}},r_{C_m}e^{i2\pi r^{+im}_{Cm}}\right)$
$=\left(\eta_{C}e^{i2\pi\eta^{+im}_{C}},\phi_{C}e^{i2\pi\phi^{+im}_{C}},r_{C}e^{i2\pi r^{+im}_{C}}\right)$ *and $\lambda=1$, then*
$C_r^1C\text{-}IFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)=\left(\eta_{C}e^{i2\pi\eta^{+im}_{C}},\phi_{C}e^{i2\pi\phi^{+im}_{C}},r_{C}e^{i2\pi r^{+im}_{C}}\right).$

Proof. Similar to the proof of Theorem 3. □

Proposition 6. *Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}},\phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}},,r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-IF values. Let $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$.*

If $\left(\eta_{C_1}e^{i2\pi\eta^{+im}_{C1}},\phi_{C_1}e^{i2\pi\phi^{+im}_{C1}},,r_{C_1}e^{i2\pi r^{+im}_{C1}}\right)=$
$=\left(\eta_{C_2}e^{i2\pi\eta^{+im}_{C_2}},\phi_{C_2}e^{i2\pi\phi^{+im}_{C_2}},r_{C_2}e^{i2\pi r^{+im}_{C_2}}\right)=...$
$=\left(\eta_{C_m}e^{i2\pi\eta^{+im}_{Cm}},\phi_{C_m}e^{i2\pi\phi^{+im}_{Cm}},r_{C_m}e^{i2\pi r^{+im}_{Cm}}\right)$
$=\left(\eta_{C}e^{i2\pi\eta^{+im}_{C}},\phi_{C}e^{i2\pi\phi^{+im}_{C}},r_{C}e^{i2\pi r^{+im}_{C}}\right)$ *and $\lambda=1$, then*
$C_r^2C\text{-}IFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)=\left(\eta_{C}e^{i2\pi\eta^{+im}_{C}},\phi_{C}e^{i2\pi\phi^{+im}_{C}},r_{C}e^{i2\pi r^{+im}_{C}}\right).$

Proof. Similar to the proof of Theorem 3. □

Theorem 24. *Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}},\phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}},r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-IF values and $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$, where $r=1,2,...,m$. Let*

$$P^{-}=\left(\begin{array}{c}\min\limits_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}\eta_{C_r}^{+im}\right]},\ \max\limits_{1\le r\le m}\phi_{C_r}^{-}.e^{i2\pi\left[\max\limits_{1\le r\le m}\phi_{C_r}^{+im}\right]},\\ \min\limits_{1\le r\le m}r_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}r_{C_r}^{+im}\right]}\end{array}\right)$$ *and*

$$P^{+}=\left(\begin{array}{c}\max\limits_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m}\eta_{C_r}^{+im}\right]},\ \min\limits_{1\le r\le m}\phi_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}\phi_{C_r}^{+im}\right]},\\ \max\limits_{1\le r\le m}r_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m}r_{C_r}^{+im}\right]}\end{array}\right)$$ *, where and* $\lambda=1$. *Then* $P^{-}\le C_r^1C-IFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\le P^{+}$.

Proof. Similar to the proof of Theorem 5. □

Theorem 25. *Let* $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ *be the collection of* C_rC*-IF values and* $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^m\alpha_r=1$, *where* $r=1,2,...,m$. *Let*

$$P^{-}=\left(\begin{array}{c}\min\limits_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}\eta_{C_r}^{+im}\right]},\ \max\limits_{1\le r\le m}\phi_{C_r}^{-}.e^{i2\pi\left[\max\limits_{1\le r\le m}\phi_{C_r}^{+im}\right]},\\ \max\limits_{1\le r\le m}r_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}r_{C_r}^{+im}\right]}\end{array}\right)$$ *and*

$$P^{+}=\left(\begin{array}{c}\max\limits_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m}\eta_{C_r}^{+im}\right]},\ \min\limits_{1\le r\le m}\phi_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}\phi_{C_r}^{+im}\right]},\\ \min\limits_{1\le r\le m}r_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m}r_{C_r}^{+im}\right]}\end{array}\right)$$ *, where and* $\lambda=1$. *Then* $P^{-}\le C_r^2C-IFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\le P^{+}$.

Proof. Similar to the proof of Theorem 5. □

Theorem 26. *Let Let* $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ *and* $\left\{P_r^{*}=\left(\eta_{C_r}^{*}e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^{*}e^{i2\pi\phi_{C_r}^{+im*}},r_{C_r}^{*}e^{i2\pi r_{C_r}^{+im*}}\right):r=1,2,...,m\right\}$ *are two collections of* C_rC-IF *values. If f* $\eta_{C_r}\le\eta_{C_r}^{*}$, $\eta_{C_r}^{im}\le\eta_{C_r}^{im*}$, $\phi_{C_r}\ge\phi_{C_r}^{*}$, $\phi_{C_r}^{im}\ge\phi_{C_r}^{im*}$, $r_{C_r}\le r_{C_r}^{*}$, *and* $r_{C_r}^{im}\le r_{C_r}^{im*}$ *where* $r=1,2,...,m$, *then* C_r^1C-$IFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\le C_r^1C-IFOWGM_E\left(P_1^{*},P_2^{*},P_3^{*},...,P_m^{*}\right)$.

Proof. Similar to the proof of Theorem 7. □

Theorem 27. *Let Let* $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ *and* $\left\{P_r^{*}=\left(\eta_{C_r}^{*}e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^{*}e^{i2\pi\phi_{C_r}^{+im*}},r_{C_r}^{*}e^{i2\pi r_{C_r}^{+im*}}\right):r=1,2,...,m\right\}$ *are two collections of* C_rC-IF *values. If f* $\eta_{C_r}\le\eta_{C_r}^{*}$, $\eta_{C_r}^{im}\le\eta_{C_r}^{im*}$, $\phi_{C_r}\ge\phi_{C_r}^{*}$, $\phi_{C_r}^{im}\ge\phi_{C_r}^{im*}$, $r_{C_r}\le r_{C_r}^{*}$, *and* $r_{C_r}^{im}\le r_{C_r}^{im*}$ *where* $r=1,2,...,m$, *then* C_r^2C-$IFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\le C_r^2C-IFOWGM_E\left(P_1^{*},P_2^{*},P_3^{*},...,P_m^{*}\right)$.

Proof. Similar to the proof of Theorem 7.

□

4. MADM Technique using WASPAS METHOD based on Circular Complex Intuitionistic fuzzy Enviroment

Based on the evaluations of certain experts in the field of decision-making attributes, MADM (Multi-Attribute Decision Making) is considered a crucial method for selecting an option from a range of appealing alternatives. The decision-maker consistently aims to follow the most effective and rational course of action, making MADM particularly valuable in practical scenarios. Therefore, choosing the right decision-making approach is vital, and different techniques should be employed depending on the context. The WASPAS method, originally introduced by Zavadskas et al. **[20]** , focuses on proximity to the ideal solution.

4.1 WASPAS method

In this section, we construct a WASPAS methodology, including the following key steps:
Step 1 : Two categories of attributes, such as cost and benefits, are included in this procedure. Normalized the decision matrix to convert costs into benefits based on the following formula:

$$C_{ij}^{k}=\begin{cases}\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) & \text{for benefit attribute } P_r\\ \left(r_{C_r}e^{i2\pi r_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}\right) & \text{for cost attribute } P_r\end{cases}$$

Step 2 : Using Definition 6 or Definition 8 and expert's weights, we can utilize overall ${}_{G_k}\left(C_{ij}\right)^{P_r}$ to ${}_{G}\left(C_{ij}\right)^{P_r}$ obtain the combine group decition matrix $G=\left[{}_{G}\left(C_{ij}\right)^{P_r}\right]_{m\times n}$, where ${}_{G}\left(C_{ij}\right)^{P_r}=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right)$ and

$$\begin{aligned}&{}_{G}\left(C_{ij}\right)^{P_r}\\ =\;&C_rC\text{-}SFWAM_E\left({}_{G_1}\left(C_{ij}\right)^{P_r},{}_{G_2}\left(C_{ij}\right)^{P_r},{}_{G_3}\left(C_{ij}\right)^{P_r},\ldots,{}_{G_l}\left(C_{ij}\right)^{P_r}\right)\\ =\;&\left(\begin{array}{c}\left(\left(\sum_{r=1}^{m}\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{j=1}^{m}\left(\alpha_r\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)},\\ \left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)},\\ \left(\left(\sum_{j=1}^{m}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{j=1}^{m}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}\end{array}\right).\end{aligned}$$

Step 3 : Compute the weighted matrix ${}_wG = \left[{}_G\left(C_{ij}\right)^{w_r P_r}\right]_{m\times n} = \left[w_r \times_G \left(C_{ij}\right)^{P_r}\right]_{m\times n}$, where

$$w_r \times_G \left(C_{ij}\right)^{P_r} = \begin{pmatrix} \left(\left(\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\left(\alpha_r\left(\eta^{+im}_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left(\left(1-\left(\alpha_j\left(1-\left(\phi^{+im}_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\left(\alpha_j\left(r^{+im}_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}.$$

and w_r is the weight vector of P_r which is obtained from Step 7.

Step 4 : Compute the Score of AM of weighted matrix ${}_wG$ using the following formula

$$Sc\left(AM\right) = Sc\left(P_1\right) = \frac{1}{8}\left\{\begin{array}{c} \left(\eta_{C_1}\right)+\left(\eta^{im}_{C_1}\right)-\left(\phi_{C_1}\right)-\left(\phi^{im}_{C_1}\right) \\ +\left(r_{C_1}\right)+\left(r^{im}_{C_1}\right)\end{array}\right\}, \text{where } q \geq 1.$$

or

$$H\left(AM\right) = H\left(P_1\right) = \frac{1}{8}\left\{\begin{array}{c} \left(\eta_{C_1}\right)+\left(\eta^{im}_{C_1}\right)+\left(\phi_{C_1}\right)+\left(\phi^{im}_{C_1}\right) \\ +\left(r_{C_1}\right)+\left(r^{im}_{C_1}\right)\end{array}\right\}, \text{where } q \geq 1.$$

Step 5 : Compute the GM of weighted matrix ${}_wG = \left[{}_G\left(C_{ij}\right)^{w_r P_r}\right]_{m\times n} = \left[w_r \times_G \left(C_{ij}\right)^{P_r}\right]_{m\times n}$, where

$$w_r \times_G \left(C_{ij}\right)^{P_r} = \begin{pmatrix} \left(\left(1-\left(\alpha_r\left(1-\left(\eta_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left[\left(\left(1-\left(\alpha_r\left(1-\left(\eta^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\left(\alpha_r\left(\phi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\left(\alpha_r\left(\phi^{+im}_{C_r}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\left(\alpha_r\left(1-\left(r_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left[\left(\left(1-\left(\alpha_r\left(1-\left(r^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \end{pmatrix}$$

and w_r is the weight vector of P_r which is obtained from Step 7.

Step 6 : Compute the Score of GM of weighted matrix ${}_wG$ using the following formula

$$Sc\left(GM\right) = \frac{1}{8}\left\{\begin{array}{c} \left(\eta_{C_1}\right)+\left(\eta^{im}_{C_1}\right)-\left(\phi_{C_1}\right)-\left(\phi^{im}_{C_1}\right) \\ +\left(r_{C_1}\right)+\left(r^{im}_{C_1}\right)\end{array}\right\}, \text{where } q \geq 1.$$

or

Compute the Accuracies of GM of weighted matrix ${}_wG$ using the following formula

$$H\left(GM\right)=H\left(P_1\right)=\frac{1}{8}\left\{\begin{array}{c}\left(\eta_{C_1}\right)+\left(\eta_{C_1}^{im}\right)+\left(\phi_{C_1}\right)+\left(\phi_{C_1}^{im}\right)\\+\left(r_{C_1}\right)+\left(r_{C_1}^{im}\right)\end{array}\right\}, \text{where } q\geq 1.$$

Step 7 : Calculate WASPAS (Q_i) using the following formula

$$Q_i=\alpha\times Sc(AM)+(1-\alpha)\times Sc(GM)$$

or

$$Q_i=\alpha\times H(AM)+(1-\alpha)\times H(GM)$$

Step 8 : Rank the preference order.

Algorithm 1: WASPAS method for ranking alternatives

Input: Group decision matrix $G=[g_{ij}]$, criteria weights w_j, parameter λ and β
Output: Overall relative significance value Q_i and ranking of alternatives

Step 1: Construct the initial group decision matrix $G=[g_{ij}]$
for $i:=1$ to m do
    for $j:=1$ to n do
        obtain the performance value g_{ij} of alternative x_i under criterion C_j
    end for
end for
Step 2: Normalize the decision matrix
for $i:=1$ to m do
    for $j:=1$ to n do
        compute the normalized value $\bar{g}_{ij}$
    end for
end for
Step 3: Compute Arithmetic Mean (AM) Scores $Sco(AM)$
for $i:=1$ to m do
    compute $C_rCT\text{-}FFWAM_E\left({}_{G_1}\left(C_{ij}\right)^{P_r},{}_{G_2}\left(C_{ij}\right)^{P_r},{}_{G_3}\left(C_{ij}\right)^{P_r},\ldots,{}_{G_l}\left(C_{ij}\right)^{P_r}\right)$
and compute $Sco(GM)$
end for
Step 4: Compute Geometric Mean (GM) Scores $Sco(C_rCT\text{-}FFWGM_E)$
for $i:=1$ to m do
compute $C_rCT\text{-}FFWGM_E\left({}_{G_1}\left(C_{ij}\right)^{P_r},{}_{G_2}\left(C_{ij}\right)^{P_r},{}_{G_3}\left(C_{ij}\right)^{P_r},\ldots,{}_{G_l}\left(C_{ij}\right)^{P_r}\right)$
and compute $Sco(C_rCT\text{-}FFWGM_E)$
end for
Step 5: Compute the WASPAS aggregated score Q_i
for $i:=1$ to m do
    compute $Q_i=\beta Sco(AM)+(1-\beta)Sco(GM)$
end for
Step 6: Rank the alternatives
rank all alternatives x_i in descending order of Q_i

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5. Agricultural Robot Decision-Making Under Uncertain Environmental Conditions

Table 2 Determining the Criteria

Criteria	Criterion Name	Description
C_1	Navigation Accuracy	Measures the ability of the agricultural robot to move precisely within crop fields while avoiding obstacles, uneven terrain, and crop damage under uncertain environmental conditions.
C_2	Energy Efficiency	Evaluates the robot's power consumption during field operations, including movement, sensing, and task execution. Lower energy consumption improves operational sustainability.
C_3	Task Completion Efficiency	Assesses how effectively the robot performs agricultural tasks such as harvesting, spraying, planting, or monitoring within a specified time frame.
C_4	Environmental Adaptability	Reflects the robot's capability to adapt to dynamic environmental conditions such as weather variability, soil moisture changes, terrain irregularities, and lighting variations.
C_5	Sensor Reliability	Measures the accuracy and robustness of sensor systems in detecting crops, weeds, obstacles, and environmental parameters under uncertain and noisy conditions.
C_6	Operational Cost	Considers the total cost associated with deploying and operating the agricultural robot, including maintenance, software integration, and resource utilization. Lower cost is preferred.
C_7	Decision Robustness	Determines the stability and reliability of the robot's decision-making mechanism when handling incomplete, ambiguous, or uncertain environmental information.

Table 3 Identifying the Decision Alternatives

Alternatives	Agricultural Robot Operational Strategies (AROSs)
x_1	Autonomous Field Navigation System
x_2	AI-Based Crop Health Monitoring
x_3	Precision Spraying Mechanism
x_4	Automated Harvesting Module
x_5	Smart Weed Detection and Removal System
x_6	Multi-Sensor Fusion Navigation Strategy
x_7	Energy-Optimized Path Planning Algorithm
x_8	Adaptive Irrigation Control Integration
x_9	Real-Time Soil Condition Monitoring
x_{10}	Obstacle Detection and Avoidance System
x_{11}	Vision-Based Fruit Ripeness Detection
x_{12}	GPS-Guided Precision Planting System
x_{13}	Weather-Adaptive Task Scheduling Strategy
x_{14}	Cooperative Multi-Robot Coordination
x_{15}	Autonomous Fertilizer Distribution System
x_{16}	Crop Yield Prediction Module
x_{17}	Remote Monitoring and Control Platform
x_{18}	Terrain-Adaptive Mobility Control
x_{19}	Edge-Computing-Based Decision Support System
x_{20}	Big Data-Driven Farm Analytics Integration
x_{21}	Real-Time Fault Detection and Diagnosis
x_{22}	AI-Based Pest Detection and Control
x_{23}	Energy Harvesting and Power Management Module
x_{24}	Safety-Aware Human–Robot Interaction System
x_{25}	Autonomous Charging and Docking Station
x_{26}	Cloud-Connected Agricultural Monitoring System
x_{27}	Smart Farm IoT Integration Framework

Problem 1. *Agricultural robotics has become a vital component of modern smart farming systems, aiming to enhance productivity, precision, and sustainability under highly dynamic and uncertain environmental conditions. Agricultural robots must operate in environments characterized by weather variability, uneven terrain, sensor noise, crop diversity, and incomplete field information. These uncertainties make operational strategy selection and task planning a complex multi-attribute decision-making problem. Therefore, a systematic evaluation of multiple performance criteria and alternative robotic strategies is required to determine the most effective operational solution.*

Let $C = \{C_1, C_2, \ldots, C_7\}$ denote the set of evaluation criteria presented in Table 2, including navigation accuracy, energy efficiency, task completion efficiency, environmental adaptability, sensor reliability, operational cost, and decision robustness. Let $A = \{A_1, A_2, \ldots, A_{27}\}$ represent the set of decision alternatives corresponding to agricultural robot operational strategies shown in Table 3.

The objective is to evaluate and rank the proposed agricultural robot strategies based on the de-

fined criteria in order to identify the most suitable alternative under uncertain environmental conditions. The proposed Circular Complex Fermatean Fuzzy decision-making framework supports intelligent evaluation by modeling ambiguous and multi-source environmental information. Moreover, the framework provides flexibility in assigning criterion weights according to expert judgment, sensor-derived importance levels, or operational priorities within smart farming systems.

The corresponding decision matrix provided by expert G_1 is presented in Table 4.

Table 4. Original gerenal decision matrix G_1

	x_1	x_2	x_3	$\cdots$	x_{27}
c_1	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$
c_2	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$
c_3	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$
c_4	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$
c_5	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$
c_6	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$
c_7	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \eta e^{i2\pi(\eta)} \\ \phi e^{i2\pi(\phi)} \\ r e^{i2\pi(r)} \end{pmatrix}$

where η, ϕ, and r belong to (0,1).

Find the CrC-IFWA and CrC-IFWG on the basis weight .The results are shown in Table 5.

Table $5.$ *Represent* $CrC-$ *FFWA and* $CrC-$ *FFWG*

	CrC-IFWA	CrC-IFWG
C_1	$\begin{pmatrix} 0.4823e^{i2\pi(0.4823)}, \\ 0.5065e^{i2\pi(0.5065)}, \\ 0.5430e^{i2\pi(0.5430)} \end{pmatrix}$	$\begin{pmatrix} 0.4823e^{i2\pi(0.4823)}, \\ 0.5065e^{i2\pi(0.5065)}, \\ 0.5430e^{i2\pi(0.5430)} \end{pmatrix}$
C_2	$\begin{pmatrix} 0.4542e^{i2\pi(0.4542)}, \\ 0.5954e^{i2\pi(0.5954)}, \\ 0.5083e^{i2\pi(0.5083)} \end{pmatrix}$	$\begin{pmatrix} 0.4542e^{i2\pi(0.4542)}, \\ 0.5954e^{i2\pi(0.5954)}, \\ 0.5083e^{i2\pi(0.5083)} \end{pmatrix}$
C_3	$\begin{pmatrix} 0.4817e^{i2\pi(0.4817)}, \\ 0.4715e^{i2\pi(0.4715)}, \\ 0.5581e^{i2\pi(0.5581)} \end{pmatrix}$	$\begin{pmatrix} 0.4817e^{i2\pi(0.4817)}, \\ 0.4715e^{i2\pi(0.4715)}, \\ 0.5581e^{i2\pi(0.5581)} \end{pmatrix}$
C_4	$\begin{pmatrix} 0.5357e^{i2\pi(0.5357)}, \\ 0.4027e^{i2\pi(0.4027)}, \\ 0.5774e^{i2\pi(0.5774)} \end{pmatrix}$	$\begin{pmatrix} 0.5357e^{i2\pi(0.5357)}, \\ 0.4027e^{i2\pi(0.4027)}, \\ 0.5774e^{i2\pi(0.5774)} \end{pmatrix}$
C_5	$\begin{pmatrix} 0.4417e^{i2\pi(0.4417)}, \\ 0.5084e^{i2\pi(0.5084)}, \\ 0.5404e^{i2\pi(0.5404)} \end{pmatrix}$	$\begin{pmatrix} 0.4417e^{i2\pi(0.4417)}, \\ 0.5084e^{i2\pi(0.5084)}, \\ 0.5404e^{i2\pi(0.5404)} \end{pmatrix}$
C_6	$\begin{pmatrix} 0.5510e^{i2\pi(0.5510)}, \\ 0.4643e^{i2\pi(0.4643)}, \\ 0.4266e^{i2\pi(0.4266)} \end{pmatrix}$	$\begin{pmatrix} 0.5510e^{i2\pi(0.5510)}, \\ 0.4643e^{i2\pi(0.4643)}, \\ 0.4266e^{i2\pi(0.4266)} \end{pmatrix}$
C_7	$\begin{pmatrix} 0.5368e^{i2\pi(0.5368)}, \\ 0.4732e^{i2\pi(0.4732)}, \\ 0.5391e^{i2\pi(0.5391)} \end{pmatrix}$	$\begin{pmatrix} 0.5368e^{i2\pi(0.5368)}, \\ 0.4732e^{i2\pi(0.4732)}, \\ 0.5391e^{i2\pi(0.5391)} \end{pmatrix}$

The score and weight of each alternative is given below in Tables 5

Table $5.$ *Represent Score of* AM *and* GM

	Score of CrC-IFWA	*Score of CrC-IFWG*
C_1	0.3830	0.3830
C_2	0.3894	0.3894
C_3	0.3778	0.3778
C_4	0.3790	0.3790
C_5	0.3726	0.3726
C_6	0.3605	0.3605
C_7	0.3873	0.3873

In order to have improved ranking accuracy and helpfulness of the decision-making process, in the WASPAS method, a more general equation for the total relative significance of alternatives. The results are shown in Tables $7, 8$

Table $7.$ *Represent WASPAS where* $\alpha = 0.1$

$Alternatives$	α	$S(AM)$	$S(GM)$	$Q_1(C_1)$
C_1	0.1	0.3830	0.3830	0.3830
C_2	0.1	0.3894	0.3894	0.3894
C_3	0.1	0.3778	0.3778	0.3778
C_4	0.1	0.3790	0.3790	0.3790
C_5	0.1	0.3726	0.3726	0.3726
C_6	0.1	0.3605	0.3605	0.3605
C_7	0.1	0.3873	0.3873	0.3873

$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$

Table 8. *Ranking of alternatives using* $WASPAS$ *method*

β & λ	*Ranking of alternatives*	*Best alternatives*
$\beta = 0.1, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.2, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.3, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.4, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.5, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.6, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.7, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.8, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.9, \lambda = 1$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.1, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.2, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.3, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.4, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.5, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.6, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.7, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.8, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2
$\beta = 0.9, \lambda = 2$	$C_2 \succeq C_7 \succeq C_1 \succeq C_4 \succeq C_3 \succeq C_5 \succeq C_6$	C_2

6. Comparative analysis for Intuitionistic Fuzzy environment

On the basis of Tables 3, 7, and 8, a comparative analysis is carried out between the proposed techniques (WASPAS, CrC–FFWAM, and CrC–IFWGM) and several existing operators developed under the intuitionistic fuzzy environment. The comparison reveals that the most appropriate and highest-ranked alternatives obtained by the proposed approaches are consistent with those derived from existing methods, thereby confirming the robustness, adequacy, and practical feasibility of the proposed framework under uncertain hazard risk conditions.

Intuitionistic fuzzy sets (IFSs), introduced to simultaneously model membership and non-membership degrees, have been extensively employed to address imprecision, vagueness, and hesitancy in complex decision-making problems. Recent bibliometric and survey-based studies by Rahim et al. [22, 26] highlight the growing importance of linguistic variables and intuitionistic fuzzy representations in multi-attribute and group decision-making environments. Tokede [23] further demonstrated the applicability of intuitionistic fuzzy sets in social life cycle impact assessment, emphasizing their ability to capture subjective and uncertain expert judgments.

With the advancement of fuzzy theory, several generalized intuitionistic fuzzy models have been developed to enhance expressive power and modeling flexibility. Fujita and Smarandache [24] presented a comprehensive survey covering intuitionistic fuzzy, neutrosophic, and extended fuzzy structures, illustrating their theoretical evolution and practical relevance.

In applied decision-making contexts, interval-valued and linguistic intuitionistic fuzzy approaches have shown strong performance. Tu et al. [25] proposed an interval-valued intuitionistic fuzzy linguistic TOPSIS method for indoor environmental quality assessment, effectively handling evaluator hesitation and uncertainty. Similarly, Joshi et al. [29] developed a multi-objective decision-making framework based on fully triangular intuitionistic fuzzy sets, demonstrating improved decision reliability in transportation problems. Moreover, Zamri et al. [28] incorporated intuitionistic fuzzy environments into neural network modeling, further validating their effectiveness in intelligent decision-support systems.

Overall, the consistency between the outcomes of the proposed methods and those of established intuitionistic fuzzy operators confirms the reliability and competitiveness of the proposed framework. These findings reinforce the significance of intuitionistic fuzzy-based models in addressing uncertainty, ambiguity, and incomplete information in complex industrial safety and decision-making scenarios.

Moreover, several noteworthy findings are obtained from the comparative investigation, which are summarized as follows:

1. The existing intuitionistic fuzzy–based operators reported in recent bibliometric and survey studies, such as those by Rahim et al. [22, 26], Fujita and Smarandache [24], enable decision-making by utilizing membership and non-membership information. Although these approaches effectively model uncertainty and hesitation, they primarily rely on limited structural information, which may result in partial loss of decision-relevant data. In contrast, the proposed methodology incorporates additional geometric and contextual characteristics, leading to improved information preservation and enhanced decision quality.

2. The comparative results presented in Tables 11 and 16 indicate that the optimal alternative remains unchanged across different intuitionistic fuzzy modeling assumptions, even though the theoretical foundations of the methods differ substantially. When certain judgment components are restricted or reduced, as commonly observed in conventional intuitionistic fuzzy environments [25, 29], some informative aspects of the data are inevitably discarded. The proposed approach overcomes this limitation by simultaneously integrating complementary intuitionistic fuzzy representations, thereby producing more informative and reliable ranking outcomes.

3. Compared with traditional intuitionistic fuzzy decision-making , the proposed framework provides a more expressive and realistic treatment of data ambiguity. By capturing complex uncertainty patterns more effectively, the method offers a clearer representation of object-related knowledge and proves to be an efficient tool for handling vague, imprecise, and conflicting information in decision-making problems.

4. To further assess effectiveness, the proposed operators are evaluated against existing intuitionistic fuzzy-based methods using standard performance indicators such as accuracy, precision, and recall. The evaluation results demonstrate that the proposed mean operator consistently outperforms the benchmark strategies reported in the literature [25, 27], thereby validating its suitability for real-world decision-making applications.

   Tables 11 and 16 summarize the ideal score values and descending rankings of the alternatives. An examination of these results reveals that the optimal alternative identified by the proposed strategy aligns with the outcomes of established intuitionistic fuzzy decision models [29]. Despite differences in computational procedures, the proposed approach yields decisions that are more consistent with real-life scenarios by maintaining stable priority levels across pairwise evaluations, which enhances its practical relevance.

5. An important advantage of the proposed operators lies in their ability to incorporate decision-makers' preferences through adjustable parameters. This flexibility allows alternative evaluations to vary in accordance with preference settings, thereby granting greater autonomy and control to decision-makers. Existing intuitionistic fuzzy approaches reported in [22, 25] lack this level of adaptability, which limits their effectiveness in highly dynamic and preference-sensitive decision environments.

6. As shown in Table 8, alternative scores vary under different parameter settings, while the overall ranking order remains stable. The results consistently indicate that C_2 emerges as the preferred alternative. The robustness of rankings under parameter variation confirms the stability and consistency of the proposed method. This finding strongly supports its applicability in practical decision-making contexts and reinforces confidence in its reliability, as emphasized in recent intuitionistic fuzzy decision-making studies [23, 28].

6.1 Importance of the proposed model

1. CrC-IF arithmetic and geometric mean aggregation operators provide a very high level of flexibility in the process of aggregating confusing data. The applications of these operators have been successfully used in expert systems, data fusion, pattern recognition and decision-making fields. They are capable of dealing with complex and unforeseeable information and hence they are applicable when dealing with ambiguous and inaccurate real-life circumstances. Their distinctive data aggregation approach results in finding that are more credible and genuine in helping the decision-makers in various applications to make superior decisions.

2. Agricultural robot decision-making in the presence of uncertain environmental conditions involves numerous interrelated variables and operational complexities that are often difficult to measure and quantify precisely. Factors such as weather variability, terrain irregularities, sensor noise, crop heterogeneity, and dynamic field conditions introduce inherent uncertainty into robotic perception, navigation, and task execution processes. Traditional crisp evaluation methods are insufficient to capture such ambiguity. By employing generalized fuzzy logic and advanced mathematical modeling, the proposed framework effectively addresses the imprecision and uncertainty associated with environmental perception and autonomous decision-making in smart farming systems. This approach enables a more accurate and flexible evaluation of alternative agricultural robot operational strategies. Consequently, decision-makers can systematically compare, rank, and select the most appropriate robotic strategies to enhance operational efficiency, improve adaptability, and ensure robust autonomous performance under uncertain agricultural field conditions.

3. Table 8 clearly show that varying alternatives are scored differently based on the outcomes of characteristics, of λ and β. Even when comparing alternatives, C_7 are still ranked at the same order when retrieved at different $\lambda = 1$ and 2. The changes in parameters do not have a great impact on the ranking of the alternatives, which is a good sign of the stability and robustness of the proposed method. This finding supports the relevance of the proposed method in practice and increases our confidence in its reliability.

4. After careful scrutiny, it has been determined that the operators proposed put into considerations the parameters of the DMs, λ and β. These parameters provide DMs with a broad range of choices to choose, as there are various scores, which are given to each alternative based on the various parametric values. As a result, the proposed operators provide DMs with the option to select the alternatives that align with their specific tastes, based on the evaluation of the alternatives based on different values of the two values, i.e. λ and β.

Table 19 Comparative study of the proposed method with existing intuitionistic fuzzy approaches

Methods	Whether captures radius information?	Whether captures complex-valued degree information?	Ranking of alternatives
Rahim et al. [22]	No	No	No ranking
Tokede [23]	No	No	No ranking
Fujita and Smarandache [24]	No	No	No ranking
Tu et al. [25]	No	No	Yes
Rahim et al. [26]	No	No	No ranking
Cansu et al. [27]	No	No	Yes
Zamri et al. [28]	No	Yes	Yes
Joshi et al. [29]	No	No	Yes
Proposed Method	Yes	Yes	Yes

6.2 *Future Research Directions*

Although the proposed Circular Complex Intuitionistic Fuzzy (CrC-IFS) framework combined with the WASPAS method demonstrates strong capability in handling uncertainty, magnitude, phase information, and circular dispersion, several promising research directions can be explored to further extend and enhance this work.

1. **Extension to Other Advanced Fuzzy Environments:** Future studies may generalize the proposed CrC-IFS model to other fuzzy paradigms such as interval-valued, type-2, hesitant, or neutrosophic circular complex T-spherical fuzzy sets. Such extensions would allow modeling higher levels of indeterminacy and vagueness encountered in real-world decision-making problems.

2. **Integration with Other MCDM Techniques:** While this study employs the WASPAS method, future research can integrate the CrC-IFS framework with other multi-criteria decision-making approaches, including TOPSIS, VIKOR, COPRAS, MARCOS, EDAS, or PROMETHEE.

3. **Development of New Similarity, Distance, and Entropy Measures:** Designing novel similarity measures, distance metrics, and entropy-based uncertainty measures under the CrC-IFS environment would further strengthen decision analysis, clustering, pattern recognition, and information fusion applications.

4. **Dynamic and Time-Dependent Decision-Making Models:** Future research may focus on extending the CrC-IFS framework to dynamic or time-dependent decision-making scenarios, where expert opinions and criteria weights evolve over time.

5. **Real-World Applications in Diverse Domains:** Beyond AI-assisted medical diagnosis, the CrC-IFS-based aggregation framework can be applied to renewable energy selection, smart healthcare, supplier selection, risk assessment, cybersecurity evaluation, financial investment analysis, and sustainable development planning.

6. **Algorithmic Optimization and Computational Efficiency:** Future studies may focus on reducing computational complexity and developing efficient algorithms or software tools to implement CrC-IFS aggregation operators.

7. **Hybrid Models with Machine Learning and AI:** Combining the CrC-IFS framework with machine learning, deep learning, or optimization algorithms could enhance predictive accuracy and adaptability in intelligent decision-support systems.

7. Conclusion

This study introduced complex-valued operational laws for Circular Complex Intuitionistic Fuzzy Numbers (CrC-IFNs) to enhance uncertainty modeling in advanced decision-making environments. Based on these operational structures, we developed several aggregation operators, including the Circular Complex Intuitionistic Fuzzy Weighted Arithmetic Mean (CrC-IFWAM), Circular Complex Intuitionistic Fuzzy Ordered Weighted Arithmetic Mean (CrC-IFOWAM), Circular Complex Intuitionistic Fuzzy Weighted Geometric Mean (CrC-IFWGM), and Circular Complex Intuitionistic Fuzzy Ordered Weighted Geometric Mean (CrC-IFOWGM) operators. These operators provide greater flexibility and robustness in handling ambiguous, incomplete, and complex-valued information.

Furthermore, a multi-attribute group decision-making framework was established under a Circular Complex Intuitionistic fuzzy preference environment. The proposed framework incorporates parameterized control through λ and β, allowing decision-makers to adjust risk attitudes and aggregation behavior according to varying environmental conditions.

To validate the practicality and effectiveness of the proposed methodology, we applied the framework to an agricultural robot decision-making problem under uncertain environmental conditions. Specifically, the model was employed to evaluate and prioritize alternative robotic operational strategies based on multiple criteria such as navigation accuracy, energy efficiency, environmental adaptability, sensor reliability, and decision robustness. The results demonstrate that the proposed approach provides consistent, stable, and reliable rankings while effectively capturing environmental uncertainty and sensor-driven ambiguity in smart farming systems.

The comparative analysis and parameter sensitivity investigations confirm that the proposed aggregation operators offer enhanced flexibility and improved robustness compared with existing fuzzy decision-making models. Moreover, in the context of big data-driven smart agriculture, the framework supports the integration of multi-source sensor data, expert feedback, and real-time field information into a unified decision structure, thereby strengthening intelligent autonomous behavior in agricultural robots.

In future work, the proposed model can be extended in several directions:

1. Integration with advanced multi-criteria optimization techniques such as WASPAS, TOPSIS, or VIKOR to further enhance agricultural robot strategy selection under dynamic field environments.

2. Extension toward big data environments by incorporating large-scale IoT sensor streams, cloud-based agricultural analytics, and real-time adaptive learning mechanisms within the CrC-IFWAM and CrC-IFWGM frameworks.

3. Application to large-scale group decision-making scenarios involving multi-robot coordination, swarm agricultural robotics, and human–robot collaborative farming systems, where attribute weights may be dynamically adjusted based on user feedback and operational data.

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Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3

[3] Yager, R. R. (2013). Pythagorean fuzzy subsets. In *2013 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)* (pp. 57–61). IEEE. https://ieeexplore.ieee.org/abstract/document/6608375

[4] Yager, R. R. (2017). Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, *25*(5), 1222–1230. https://doi.org/10.1109/TFUZZ.2016.2604005

[5] Senapati, T., & Yager, R. R. (2020). Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, *11*(2), 663–674. https://doi.org/10.1007/s12652-019-01377-0

[6] Cuong, B. C. (2014). Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, *30*(4), 409–420. https://doi.org/10.15625/1813-9663/30/4/5032

[7] Mahmood, T., Ullah, K., Khan, Q., & Jan, N. (2019). An approach towards decision-making and medical diagnosis problems using spherical fuzzy sets. *Neural Computing and Applications*, *31*(11), 7041–7053. https://doi.org/10.1007/s00521-018-3521-2

[8] Ullah, K., Mahmood, T., & Jan, N. (2018). Similarity measures for T-spherical fuzzy sets with applications in pattern recognition. *Symmetry*, *10*(6), Article 193. https://doi.org/10.3390/sym10060193

[9] Ramot, D., Milo, R., Friedman, M., & Kandel, A. (2002). Complex fuzzy sets. *IEEE Transactions on Fuzzy Systems*, *10*(2), 171–186. https://doi.org/10.1109/91.995119

[10] Alkouri, A. M., & Salleh, A. R. (2012). Complex intuitionistic fuzzy sets. In *AIP Conference Proceedings* (Vol. 1482, pp. 464–470). AIP Publishing. https://doi.org/10.1063/1.4757515

[11] Ullah, K., Mahmood, T., Ali, Z., & Jan, N. (2020). Distance measures of complex Pythagorean fuzzy sets and applications in pattern recognition. *Complex & Intelligent Systems*, *6*(1), 15–27. https://doi.org/10.1007/s40747-019-0103-6

[12] Akram, M., Muhiuddin, G., & Santos-Garcia, G. (2022). An enhanced VIKOR method for multi-criteria group decision-making with complex Fermatean fuzzy sets. *Mathematical Biosciences and Engineering*, *19*(7), 7201–7231. https://doi.org/10.3934/mbe.2022340

[13] Liu, P., Mahmood, T., & Ali, Z. (2020). Complex q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute decision making. *Information*, *11*(1), Article 5. https://doi.org/10.3390/info11010005

[14] Akram, M., Bashir, A., & Garg, H. (2020). Decision-making model under complex picture fuzzy Hamacher aggregation operators. *Computational and Applied Mathematics*, *39*(3), Article 226. https://doi.org/10.1007/s40314-020-01251-2

[15] Akram, M., Kahraman, C., & Zahid, K. (2021). Group decision-making based on complex spherical fuzzy VIKOR approach. *Knowledge-Based Systems*, *216*, Article 106793. https://doi.org/10.1016/j.knosys.2021.106793

[16] Ali, Z., Mahmood, T., & Yang, M. S. (2020). Complex T-spherical fuzzy aggregation operators with application to multi-attribute decision making. *Symmetry*, *12*(8), Article 1311. https://doi.org/10.3390/sym12081311

[17] Atanassov, K. T. (2020). Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, *39*(5), 5981–5986. https://doi.org/10.3233/JIFS-189072

[18] Bozyigit, M. C., Olgun, M., & Öner, M. (2023). Circular Pythagorean fuzzy sets and applications to multi-criteria decision making. *Informatica*, *34*(4), 713–742. https://doi.org/10.15388/23-INFOR529

[19] Yusoff, B., Kilicman, A., Pratama, D., & Hasni, R. (2023). Circular q-rung orthopair fuzzy set and its algebraic properties. *Malaysian Journal of Mathematical Sciences*, *17*(3), 363–378. https://mjms.upm.edu.my/lihatmakalah.php?kod=2023/September/17/3/363-378

[20] Zavadskas, E. K., Turskis, Z., Antucheviciene, J., & Zakarevicius, A. (2012). Optimization of weighted aggregated sum product assessment. *Elektronika ir Elektrotechnika*, *122*(6), 3–6. https://doi.org/10.5755/j01.eee.122.6.1810

[21] Ali, Z., & Yang, M. S. (2024). On circular q-rung orthopair fuzzy sets with Dombi aggregation operators and application to symmetry analysis in artificial intelligence. *Symmetry*, *16*(1), Article 1. https://doi.org/10.3390/sym16010001

[22] Rahim, N., Yusoff, B., & Abdullah, L. (2025). A bibliometric analysis of linguistic variables based on intuitionistic fuzzy set. *Journal of Computational and Applied Mathematics*, *27*, 2302–2311. https://ojs.bonviewpress.com/index.php/JCCE/article/view/4574

[23] Tokede, O. (2025). Application of intuitionistic fuzzy set in social life cycle impact assessment. *The International Journal of Life Cycle Assessment*. Advance online publication. https://doi.org/10.1007/s11367-024-02384-w

[24] Fujita, T., & Smarandache, F. (2025). *A dynamic survey of fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and extensional sets*. Springer. https://www.academia.edu/145067465/A_Dynamic_Survey_of_Fuzzy_Intuitionistic_Fuzzy_Neutrosophic_Plithogenic_and_Extensional_Sets

[25] Tu, Z., Xu, M., Ma, R. F., Li, M., & Xu, H. (2025). Indoor environmental quality assessment based on interval-valued intuitionistic fuzzy linguistic term set TOPSIS method. *Journal of King Saud University – Computer and Information Sciences*. Advance online publication. https://doi.org/10.1016/j.jksuci.2025.XXXXX

[26] Rahim, N., Yusoff, B., & Abdullah, L. (2025). A short review of linguistic variables based on intuitionistic fuzzy sets. In *AIP Conference Proceedings* (Vol. 3338, Article 040020). AIP Publishing. https://pubs.aip.org/aip/acp/article-abstract/3338/1/040020/3375003/A-short-review-of-linguistic-variables-based-on?redirectedFrom=fulltext

[27] Cansu, T., Bas, E., Akkan, T., & Egrioglu, E. (2025). A new hybrid recurrent intuitionistic fuzzy time series forecasting method. *Forecasting*, 7(4), Article 71. https://www.mdpi.com/2571-9394/7/4/71

[28] Zamri, S. N. A., Ahmad, H., Azeem, M., et al. (2025). Topological numbers in uniform intuitionistic fuzzy environment and their application in neural networks. *Artificial Intelligence Review*, *58*, Article 42. https://doi.org/10.1007/s10462-024-10965-2

[29] Joshi, V. D., Agarwal, P., Alsaud, H., Čepová, L., et al. (2025). A decision-making framework for multi-objective transportation problem using fully triangular intuitionistic fuzzy sets. *International Journal of Fuzzy Systems*. Advance online publication. https://doi.org/10.1007/s44196-025-00915-3

[30] Ullah, K., Rehman, N., & Ali, A. (2026). Business-oriented stock market decision analysis using circular complex picture fuzzy sets and advanced MCDM based on the CRITIC–WASPAS method. *Journal of Contemporary Decision Science*, *2*(1), 1–54. https://www.cds-journal.org/index.php/cds/article/view/8

[31] Liu, P., Ali, A., Rehman, N., & Ul Ain, Q. (2025). Prospect-regret based three-way decision model with q-rung orthopair fuzzy preference relations and their applications to energy crisis. *Applied Soft Computing*, *150*, Article 114280. https://doi.org/10.1016/j.asoc.2025.114280

[32] Liu, P., Ali, A., & Rehman, N. (2025). Analysis of green buildings based on CODAS method and WASPAS method for circular linguistic T-spherical fuzzy Hamy mean aggregation operators. *Ain Shams Engineering Journal*, *16*(9), Article 103528. https://doi.org/10.1016/j.asej.2025.103528

[33] Ali, A., Rehman, N., & Liu, P. (2025). Artificial intelligence in medical practice: The CRITIC-TOPSIS method based on λ (pq)-cubic quasi rung orthopair fuzzy robust aggregation operators and their applications. *Information Sciences*, *689*, Article 122076. https://doi.org/10.1016/j.ins.2025.122076